MATH 54 FINAL EXAM (PRACTICE 2)

PROFESSOR PAULIN

DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.

CALCULATORS ARE NOT PERMITTED

YOU MAY USE YOUR OWN BLANK PAPER FOR ROUGH WORK

REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT

THIS EXAM WILL BE ELECTRONICALLY SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED. YOU MAY WRITE SOLUTIONS ON THE BLANK PAGE AT THE BACK BUT BE SURE TO CLEARLY LABEL THEM

Name and section: ____

GSI's name: ____

This exam consists of 10 questions. Answer the questions in the spaces provided.

1. (25 points) (a) Are the following matrices row equivalent?

$$\begin{pmatrix} 2 & 0 & 4 & 0 \\ 0 & 1 & 3 & 0 \\ 1 & -1 & -1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 2 & 1 \\ 0 & 2 & -2 & 2 \\ 2 & 0 & 4 & 3 \end{pmatrix}$$

Solution:

(b) What are the dimensions of the null and column spaces of the above matrices?

Solution:

2. (25 points) Do the following vectors span $\mathbb{R}^3$?

$$\begin{pmatrix}1\\2\\0\end{pmatrix}, \begin{pmatrix}1\\0\\-1\end{pmatrix}, \begin{pmatrix}2\\2\\-1\end{pmatrix}, \begin{pmatrix}3\\2\\-2\end{pmatrix}$$

If a matrix has three of these vectors as columns, can it be invertible? Carefully justify your answers.

Solution:

3. (25 points) (a) Let $T : V \to W$ be a linear transformation between two vector spaces. Define the kernel of T, $Ker(T)$. Show that $Ker(T)$ is a subspace of V. You may assume that $T(\underline{0}_V) = \underline{0}_W$

Solution:

(b) Does there exist a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$, such that

$Ker(T) = \{\begin{pmatrix} x \\ x+1 \end{pmatrix}\}$ where x is any real number?

Solution:

4. (25 points) Let T be the following linear transformation:

$$\begin{aligned} T : \mathbb{P}_2(\mathbb{R}) &\rightarrow \mathbb{P}_2(\mathbb{R}) \\ p(x) &\mapsto p'(x) + p(x) \end{aligned}$$

Does there exist a basis $B \subset \mathbb{P}_2(\mathbb{R})$ such that $A_{B,B}$ is diagonal? Justify your answer. Hint: Think about the possible degrees of the polynomials in B.

Solution:

5. (25 points) Let $C[-1,1]$ be the inner product space of real-valued functions on the closed interval $[-1,1]$, such that

$$< f, g >= \int_{-1}^{1} f(x)g(x)dx.$$

Find an orthogonal basis for $W = Span(1, x^2, x^4)$.

Solution:

6. (25 points) Determine all least-squares solutions to the following linear system:

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{pmatrix} \underline{x} = \begin{pmatrix} 1 \\ 3 \\ 8 \\ 2 \end{pmatrix}$$

Solution:

7. (25 points) Find orthonormal bases of $\mathbb{R}^2$, B and C, such that if

$$A = \begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$$

then $A_{B,C}$ is diagonal with non-negative entries.

Solution:

Solution (continued) :

8. (25 points) Find a general solution to the following differential equation

$$y'' - y = t\cos(t) + \sin(t)$$

Solution:

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Solution (continued) :

9. (25 points) Find a general solution to

$$\underline{x}'(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \underline{x}(t).$$

Solution:

Solution (continued) :

10. (25 points) Calculate the cosine Fourier series of the function $f(x) = e^x$, on the interval $[0, \pi]$.

Solution:

END OF EXAM