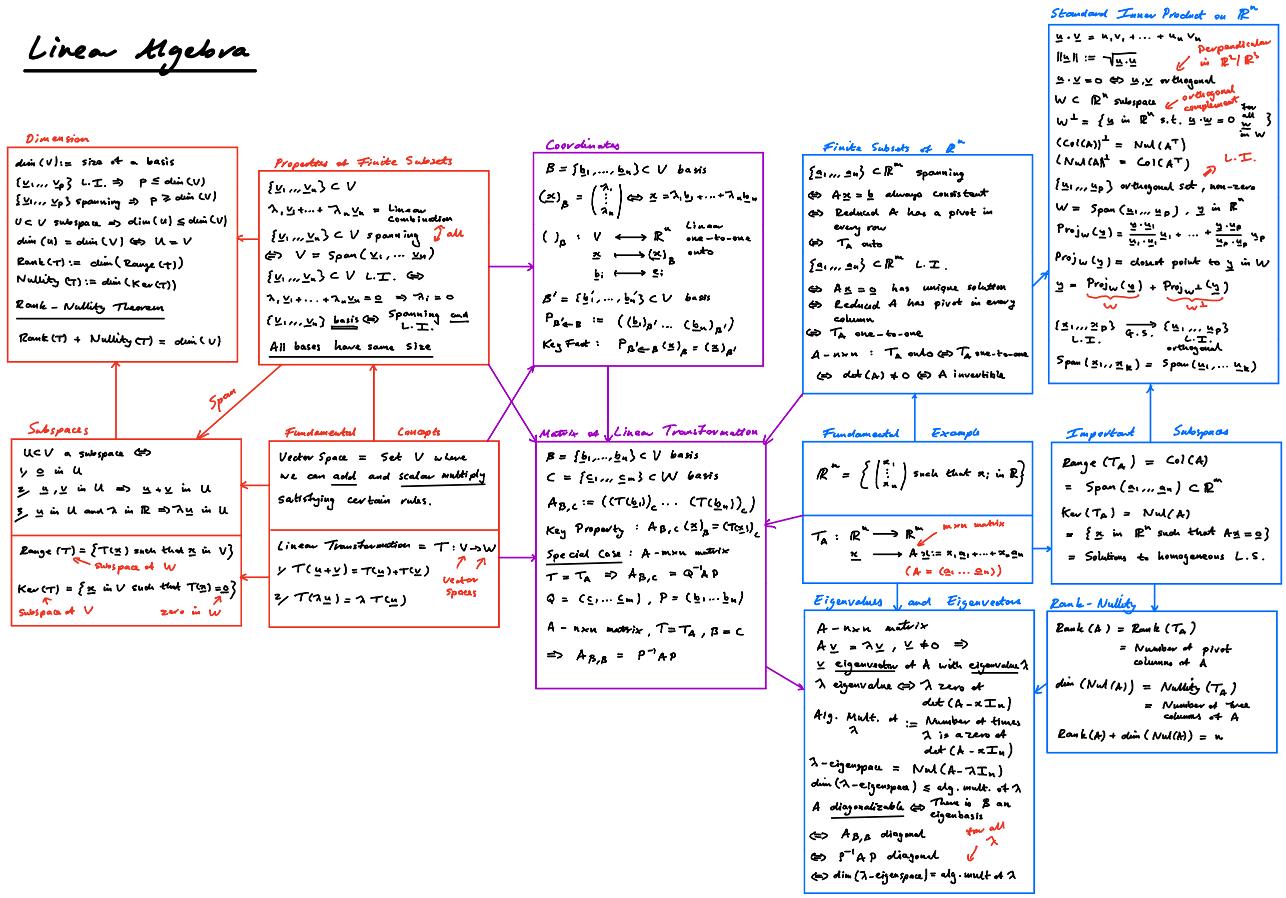
Linear Algebra



Example Find bases for $Col(A)$ and $Nul(A)$, $A = \begin{pmatrix} 1 & -1 & 0 & 1 \\ 1 & -1 & 1 & 1 \\ 2 & -2 & 0 & 2 \end{pmatrix}$.

$$\begin{pmatrix} 1 & -1 & 0 & 1 \\ 1 & -1 & 1 & 1 \\ 2 & -2 & 0 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} \boxed{1} & -1 & 0 & 1 \\ 0 & 0 & \boxed{1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \Rightarrow \left\{ \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\} \text{ basis for } Col(A)$$

$$\left(\begin{array}{cccc|c} \boxed{1} & -1 & 0 & 1 & 0 \\ 0 & 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow Nul(A) = \left\{ \begin{pmatrix} x_2 - x_4 \\ x_2 \\ 0 \\ x_4 \end{pmatrix} \right\} = \left\{ x_2 \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

$$\Rightarrow \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \right\} \text{ basis for } Nul(A)$$

Example $V = \mathbb{P}_2(\mathbb{R})$, $\mathcal{B} = \{1, 1+x, 1+x-x^2\}$

$\mathcal{B}' = \{x^2, x-1, 1\}$

$$P_{\mathcal{B}'\leftarrow\mathcal{B}} = \left((\underline{b}_1)_{\mathcal{B}'} \;\; (\underline{b}_2)_{\mathcal{B}'} \;\; (\underline{b}_3)_{\mathcal{B}'} \right)$$

$$= \left((1)_{\mathcal{B}'} \;\; (1+x)_{\mathcal{B}'} \;\; (1+x-x^2)_{\mathcal{B}'} \right)$$

$$1 = \lambda_1 x^2 + \lambda_2 (x-1) + \lambda_3 1 \Rightarrow \lambda_1 = 0, \lambda_2 = 0, \lambda_3 = 1$$

$$1+x = \lambda_1 x^2 + \lambda_2 (x-1) + \lambda_3 1 \Rightarrow \lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 2$$

$$1+x-x^2 = \lambda_1 x^2 + \lambda_2 (x-1) + \lambda_3 1 \Rightarrow \lambda_1 = -1, \lambda_2 = 1, \lambda_3 = 2$$

$$\Rightarrow \quad P_{\mathcal{B}'\leftarrow\mathcal{B}} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 1 \\ 1 & 2 & 2 \end{pmatrix}$$

<u>Example</u> $A = \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$, $\mathcal{B} = \{ \underset{\underline{b}_1}{\underset{\shortparallel}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}}, \underset{\underline{b}_2}{\underset{\shortparallel}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}} \}$, $C = \{ \underset{\underline{c}_1}{\underset{\shortparallel}{\begin{pmatrix} -1 \\ 1 \end{pmatrix}}}, \underset{\underline{c}_2}{\underset{\shortparallel}{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}} \}$ $(T = T_A)$

$A_{\mathcal{B},C} = ?$

$$A_{\mathcal{B},C} = \left((T_A(\underline{b}_1))_C \;\; (T_A(\underline{b}_2))_C \right)$$

$$= \left((A\underline{b}_1)_C \;\; (A\underline{b}_2)_C \right)$$

$$= \left(\begin{pmatrix} 3 \\ 4 \end{pmatrix}_C \;\; \begin{pmatrix} 1 \\ 3 \end{pmatrix}_C \right)$$

$$\left(\begin{array}{cc|c} -1 & 0 & 3 \\ 1 & 1 & 4 \end{array} \right) \longrightarrow \left(\begin{array}{cc|c} 1 & 0 & -3 \\ 0 & 1 & 7 \end{array} \right) \Rightarrow \begin{pmatrix} 3 \\ 4 \end{pmatrix}_C = \begin{pmatrix} -3 \\ 7 \end{pmatrix}$$

$$\begin{pmatrix} 3 \\ 4 \end{pmatrix} = (-3)\begin{pmatrix} -1 \\ 1 \end{pmatrix} + 7\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\left(\begin{array}{cc|c} -1 & 0 & 1 \\ 1 & 1 & 3 \end{array} \right) \longrightarrow \left(\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 4 \end{array} \right) \Rightarrow \begin{pmatrix} 1 \\ 3 \end{pmatrix}_C = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$$

$$\Rightarrow \quad A_{B,C} = \begin{pmatrix} -3 & -1 \\ 7 & 4 \end{pmatrix} \quad \left(B = \Big\{ \underset{\substack{\shortparallel \\ \underline{b}_1}}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}, \underset{\substack{\shortparallel \\ \underline{b}_2}}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} \Big\}, \; C = \Big\{ \underset{\substack{\shortparallel \\ \underline{c}_1}}{\begin{pmatrix} -1 \\ 1 \end{pmatrix}}, \underset{\substack{\shortparallel \\ \underline{c}_2}}{\begin{pmatrix} 0 \\ 1 \end{pmatrix}} \Big\} \; (T = T_A) \right)$$

$$A_{B,C} = Q^{-1} A P = \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

<u>Example</u> $\quad A = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \quad B = \Big\{ \underset{\substack{\shortparallel \\ \underline{b}_1}}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}, \underset{\substack{\shortparallel \\ \underline{b}_2}}{\begin{pmatrix} 1 \\ 1 \end{pmatrix}} \Big\}, \quad T = T_A$

$A_{B,B} = ?$

$$A_{B,B} = \left((A\underline{b}_1)_B \; (A\underline{b}_2)_B \right) = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}_B \; \begin{pmatrix} 2 \\ 2 \end{pmatrix}_B \right)$$

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 0 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 \\ 2 \end{pmatrix} = 0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \Rightarrow \quad A_{B,B} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

$$A \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$A_{B,B} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$$

<u>Example</u> $\quad$ Is $A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 3 \end{pmatrix}$ diagonalizable ?

$1, 2, 3$ = eigenvalues of A

alg. mult. of $1 = 2 \geq$ dim (1-eigenspace)

$$\left.\begin{array}{l} \text{alg. mult. of } 2 = 1 \geq \dim(2\text{-eigenspace}) \geq 1 \\ \text{alg. mult of } 3 = 1 \geq \dim(3\text{-eigenspace}) \geq 1 \end{array}\right\} \text{dimensions are } 1$$

$$A - 1 \cdot I_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 & \boxed{1} & 0 & 0 \\ 0 & 0 & \boxed{1} & 1 \\ 0 & 0 & 0 & \boxed{2} \end{pmatrix} \Rightarrow \dim(1\text{-eigenspace}) = 1$$

$0 \quad 0 \quad 0 \quad 2 \qquad\qquad 0 \quad 0 \quad 0 \quad 0$

$\Rightarrow$ A not diagonalizable.

Example

$$W = \operatorname{Span}\left\{ \overset{\underline{x}_1}{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}}, \overset{\underline{x}_2}{\begin{pmatrix}1\\1\\0\\0\end{pmatrix}}, \overset{\underline{x}_3}{\begin{pmatrix}1\\1\\1\\0\end{pmatrix}} \right\} = \operatorname{Span}\left(\overset{\underline{v}_1}{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}}, \overset{\underline{v}_2}{\begin{pmatrix}0\\1\\0\\0\end{pmatrix}}, \overset{\underline{v}_3}{\begin{pmatrix}0\\0\\1\\0\end{pmatrix}} \right)$$

$$\underline{v}_1 = \underline{x}_1$$

$$\underline{v}_2 = \underline{x}_2 - \frac{\underline{x}_2 \cdot \underline{v}_1}{\underline{v}_1 \cdot \underline{v}_1}\underline{v}_1 = \begin{pmatrix}1\\1\\0\\0\end{pmatrix} - \frac{1}{1}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

$$\underline{v}_3 = \underline{x}_3 - \frac{\underline{x}_3 \cdot \underline{v}_1}{\underline{v}_1 \cdot \underline{v}_1}\underline{v}_1 - \frac{\underline{x}_3 \cdot \underline{v}_2}{\underline{v}_2 \cdot \underline{v}_2}\underline{v}_2 = \begin{pmatrix}1\\1\\1\\0\end{pmatrix} - \frac{1}{1}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} - \frac{1}{1}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$$

$$= \begin{pmatrix}0\\0\\1\\0\end{pmatrix}$$

What is distance between W and $\begin{pmatrix}3\\2\\1\\1\end{pmatrix}$

Min distance between W and $\underset{\underline{y}}{\begin{pmatrix}3\\2\\1\\1\end{pmatrix}}$

$$= \left\| \overbrace{\underline{y} - \operatorname{Proj}_W(\underline{y})}^{\operatorname{Proj}_{W^\perp}(\underline{y})} \right\|$$

$$= \left\| \begin{pmatrix}3\\2\\1\\1\end{pmatrix} - \frac{\begin{pmatrix}3\\2\\1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\0\\0\\0\end{pmatrix}}{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}\cdot\begin{pmatrix}1\\0\\0\\0\end{pmatrix}}\begin{pmatrix}1\\0\\0\\0\end{pmatrix} - \frac{\underline{y}\cdot\underline{v}_2}{\underline{v}_2\cdot\underline{v}_2}\underline{v}_2 - \frac{\underline{y}\cdot\underline{v}_3}{\underline{v}_3\cdot\underline{v}_3}\underline{v}_3 \right\|$$

$$= \left\| \begin{pmatrix}3\\2\\1\\1\end{pmatrix} - \begin{pmatrix}3\\2\\1\\0\end{pmatrix} \right\|$$

$$= \left\| \begin{pmatrix}0\\0\\0\\1\end{pmatrix} \right\| = 1$$

$$W = \operatorname{Col}\begin{pmatrix}1&1&1\\0&1&1\\0&0&1\\0&0&0\end{pmatrix} \Rightarrow W^\perp = \left(\operatorname{Col}\begin{pmatrix}1&1&1\\0&1&1\\0&0&1\\0&0&0\end{pmatrix}\right)^\perp$$

$$= \text{Nul}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \Rightarrow W^{\perp} = \text{Span}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \text{Proj}_{W^{\perp}}(\underline{y}) = \frac{\begin{pmatrix} 3 \\ 2 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}}{\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\Rightarrow \| \text{Proj}_{W^{\perp}}(\underline{y}) \| = 1$$