Midterm 1 Review

Linear System :
$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\quad \vdots \qquad\qquad\quad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

$x_1, x_2, \ldots, x_n$ = Solution to linear system

Augmented Matrix :
$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & & a_{mn} & b_m \end{array}\right)$$

Coefficient Matrix A

Key Fact: Row operations on augmented matrix alter linear system but <u>not</u> the set of solutions.

Algorithm to find general solution :

Linear System

$\downarrow$ Represent with matrix

Augmented Matrix

$\downarrow$ Row Reduction

Echelon / Reduced Echelon Augmented Matrix

- Last column is pivot $\rightarrow$ Inconsistent System
- Last column not pivot:
  - No free columns $\rightarrow$ Unique Solution
  - Free Columns $\rightarrow$ Infinitely many Solutions

Unique Solution and Infinitely many Solutions: Found by backwards substitution.

Example

$$\begin{array}{ccc} & \text{pivot position} & \\ \left(\begin{array}{ccc|c} \boxed{1} & 0 & 2 & 0 \\ 0 & \boxed{1} & 3 & 1 \end{array}\right) & \Rightarrow & \begin{array}{l} x_1 + 2x_3 = 0 \\ x_2 + 3x_3 = 1 \end{array} \end{array}$$

Pivot Columns   Free Column

$$\Rightarrow \quad \text{General Solution} = \begin{cases} -2x_3 \\ 1 - 3x_3 \\ x_3 \text{ free} \end{cases}$$

Vector/Matrix Perspective

$$\mathbb{R}^m = \left\{ \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix} \text{ such that } v_i \text{ in } \mathbb{R} \right\} \quad \text{we write } \underline{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix}$$

called vector in $\mathbb{R}^m$

Key Fact: We can add and scale vectors in $\mathbb{R}^m$.

$$\underline{u} = \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix}, \ \underline{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_m \end{pmatrix} \quad \Rightarrow \quad \underline{u} + \underline{v} := \begin{pmatrix} u_1 + v_1 \\ \vdots \\ u_m + v_m \end{pmatrix}$$

$$\lambda \text{ in } \mathbb{R} \qquad \lambda \underline{u} = \begin{pmatrix} \lambda u_1 \\ \vdots \\ \lambda u_m \end{pmatrix}$$

A – $m \times n$ matrix $\Rightarrow A = (\underline{a}_1 \ldots \underline{a}_n)$ in $\mathbb{R}^m$

Notation: $\underline{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ in $\mathbb{R}^n$

linear combination

$$\Rightarrow \quad A\underline{x} := x_1 \underline{a}_1 + x_2 \underline{a}_2 + \cdots + x_n \underline{a}_n$$

Let $\underline{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix}$ in $\mathbb{R}^m$

$\underline{x}$ is solution to $(A \mid \underline{b}) \iff A\underline{x} = \underline{b}$

$\Rightarrow (A \mid \underline{b})$ consistent $\iff$ $\underline{b}$ a linear combination of $\{\underline{a}_1, \ldots, \underline{a}_n\}$

admits a solution

$\iff \underline{b}$ in $\mathrm{Span}(\underline{a}_1, \ldots, \underline{a}_n)$

$\qquad\qquad \shortparallel$

all linear combinations

Conclusion

$\underline{b}$ in Span$(\underline{a}_1, \dots, \underline{a}_n) \iff (A \mid \underline{b})$ consistent $\iff$ Last column of reduced $(A \mid \underline{b})$ not a pivot

E.g. $\left(\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 3 \\ 0 & \boxed{1} & 2 & 4 \\ 0 & 0 & 0 & 0 \end{array}\right) \Rightarrow \begin{pmatrix} 3 \\ 4 \\ 0 \end{pmatrix}$ in Span$\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}\right)$

Not pivot

$\left(\begin{array}{ccc|c} \boxed{1} & 1 & 1 & 3 \\ 0 & \boxed{1} & 2 & 4 \\ 0 & 0 & 0 & \boxed{1} \end{array}\right) \Rightarrow \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$ <u>not</u> in Span$\left(\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}\right)$

Pivot Position in every row of reduced A $\iff$ Every row of reduced A is non-zero $\iff$ Last column of reduced $(A \mid \underline{b})$ is not pivot for any $\underline{b}$ in $\mathbb{R}^m$

$$\boxed{\text{Span}(\underline{a}_1, \dots, \underline{a}_n) = \mathbb{R}^m \iff \text{Reduced } A \text{ has pivot position in } \underline{\text{every row}}}$$

$\{\underline{a}_1, \dots, \underline{a}_n\}$ Linearly independent (L.I.):

$$\underbrace{x_1\underline{a}_1 + x_2\underline{a}_2 + \dots + x_n\underline{a}_n}_{= A\underline{x}} = \underline{0} \Rightarrow x_1 = x_2 = \dots = x_n = 0$$

homogeneous linear system

$\Rightarrow \{\underline{a}_1, \dots, \underline{a}_n\}$ L.I. $\iff A\underline{x} = \underline{0}$ has unique solution $\underline{x} = \underline{0}$

$\iff$ Reduced $(A \mid \underline{0})$ has no free columns

$\iff$ Reduced A has pivot position in every column

Fact General solution to $A\underline{x} = \underline{b}$ $=$ $\underline{v}_p$ $+$ General Solution to $A\underline{x} = \underline{0}$

Fixed particular solution

$\Rightarrow$ Consistent $A\underline{x} = \underline{b}$ has unique solution $\Leftrightarrow$ $A\underline{x} = \underline{0}$ has unique solution $\Leftrightarrow$ $\{\underline{a}_1, \ldots, \underline{a}_n\}$ L.I.

Conclusion

$$\boxed{\{\underline{a}_1, \ldots, \underline{a}_n\} \text{ L.I.} \Leftrightarrow \text{Reduced } A \text{ has pivot position in } \underline{\text{every column}}}$$

Linear Transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$

$T : \mathbb{R}^n \to \mathbb{R}^m$ linear $\Leftrightarrow$

1/ $T(\underline{x} + \underline{y}) = T(\underline{x}) + T(\underline{y})$

2/ $T(\lambda \underline{x}) = \lambda T(\underline{x})$

$A = (\underline{a}_1 \ldots \underline{a}_n)$ - $m \times n$ matrix $\Rightarrow$

$$T_A : \mathbb{R}^n \to \mathbb{R}^m$$
$$\underline{x} \mapsto A\underline{x} = x_1 \underline{a}_1 + \cdots + x_n \underline{a}_n$$

Linear

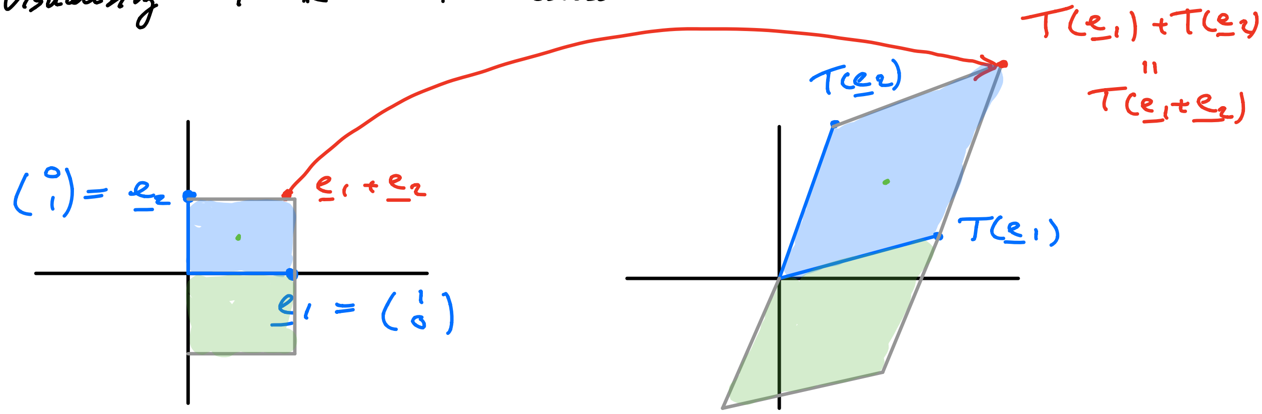
Fact : T linear $\Rightarrow$ $T(\underline{x}) = A\underline{x}$ where

$A = (T(\underline{e}_1)\ T(\underline{e}_2) \ldots T(\underline{e}_n))$ = Standard Matrix of T

$$\underline{e}_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \underline{e}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \ldots, \underline{e}_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}$$

If we know $T(\underline{e}_1), \ldots, T(\underline{e}_n)$ we know $T(\underline{x})$ for all $\underline{x}$

Visualising $T: \mathbb{R}^2 \to \mathbb{R}^2$ linear :



Range of T_A = $\{A\underline{x}\}$ = Span$(\underline{a}_1, \dots, \underline{a}_n)$

$\Rightarrow$ T_A onto $\iff$ Span$(\underline{a}_1, \dots, \underline{a}_n) = \mathbb{R}^m$

T_A one-to-one $\iff$ $T_A(\underline{x}) = \underline{b}$ has at most one solution $\iff$ $A\underline{x} = \underline{b}$ has at most one solution $\iff$ $\{\underline{a}_1, \dots, \underline{a}_n\}$ L.I.

Overview

Linear Transformations		Vectors		Linear Systems		Reduced Echelon Matrices
$\underline{b}$ in range of T_A	$\iff$	$\underline{b}$ in Span$(\underline{a}_1, \dots, \underline{a}_n)$	$\iff$	$A\underline{x} = \underline{b}$ has solution	$\iff$	Reduced $(A \mid \underline{b})$ does <u>not</u> have pivot position in last column
T_A <u>onto</u>	$\iff$	Span$(\underline{a}_1, \dots, \underline{a}_n) = \mathbb{R}^m$	$\iff$	$A\underline{x} = \underline{b}$ has solution <u>for all $\underline{b}$</u>	$\iff$	Pivot position in <u>every row</u> of reduced A
T_A <u>one-to-one</u>	$\iff$	$\{\underline{a}_1, \dots, \underline{a}_n\}$ L.I.	$\iff$	$A\underline{x} = \underline{b}$ has <u>at most</u> one solution	$\iff$	Pivot position in <u>every column</u> of reduced A.

Matrix Algebra

A, B – $m \times n$ matrices, λ in $\mathbb{R}$

$$(A+B)_{ij} := (A)_{ij} + (B)_{ij}$$ ← ijth entry

$$(\lambda A)_{ij} := \lambda (A_{ij})$$

A – $m \times n$, B – $n \times p$

$$(AB)_{ij} = (A)_{11}(B)_{1j} + (A)_{i2}(B)_{2j} + \dots + (A)_{in}(B)_{nj}$$

Key Facts:

$$T_{(A+B)} = T_A + T_B$$
$$T_{(\lambda A)} = \lambda T_A$$
$$T_{AB} = T_A \circ T_B$$

Invertible Matrices

A – $n \times n$ matrix

A invertible $\iff$ There exists B, an $n \times n$ matrix, such that $AB = BA = I_n$ = $n \times n$ identity Matrix

denoted A^{-1}, the inverse of A

Fact: A invertible $\iff$ $T_A : \mathbb{R}^n \to \mathbb{R}^n$ one-to-one and onto

$\iff$ Reduced A has pivot position in every column and every row

$\iff$ A row equivalent to $\begin{pmatrix} \blacksquare & & * \\ & \blacksquare & \\ 0 & & \ddots \;\blacksquare \end{pmatrix}$

$\iff$ Reduced A equals I_n

Crazy coincidence in $n \times n$ case: These are equivalent

Algorithm to Calculate A^{-1}

1/ Write $(A \mid I_n)$

2/ Put in reduced echelon form $(I_n \mid B)$ using row operations

3/ $B = A^{-1}$.

Determinant

determinant of A

Given A - $n \times n$ matrix, $\det(A)$ is a number which helps us determine invertibility.

Facts 1/ U upper triangular $\Rightarrow \det(U) =$ product of diagonal entries

E.g. $\det\begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 3 \\ 0 & 0 & 3 \end{pmatrix} = 1 \cdot 2 \cdot 3 = 6$

2/ A/ Switching two rows multiplies determinant by -1

B/ Adding scalar multiple of one row to another does not change determinant

3/ Square matrix in echelon form $\Rightarrow$ upper triangular

Algorithm to calculate $\det(A)$

1/ Put A in echelon form using only A/ and B/

2/ Compute product of diagonal entries (This is the determinant of)

3/ $\det(A) = (-1)^r \times$ Product of diagonal entries in echelon form

Number of row switches

<u>Fact</u> A^{-1} exists $\iff$ $\det(A) \neq 0$