Homework 7 Solutions

§11.8 Power Series

Q3/ $a_n = (-1)^n n x^n \Rightarrow$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{n}|x| \longrightarrow |x| \quad \text{as } n \to \infty$$

$$\left.\begin{array}{l} \Rightarrow \displaystyle\sum_{n=1}^{\infty} a_n \quad \text{conv} \quad \text{if} \quad |x| < 1 \\ \displaystyle\sum_{n=1}^{\infty} a_n \quad \text{div} \quad \text{if} \quad |x| > 1 \end{array}\right\} R = 1$$

$x = -1 \Rightarrow a_n = n \longrightarrow \infty$ as $n \to \infty$

$\Rightarrow$ divergent

$x = 1 \Rightarrow a_n = (-1)^n n \not\longrightarrow 0$ as $n \to \infty$

$\Rightarrow$ divergent

$\Rightarrow$ I.O.C. $= (-1, 1)$.

Q10/ $a_n = 2^n n x^n$

$$\Rightarrow \left|\frac{a_{n+1}}{a_n}\right| = 2 \cdot \frac{n+1}{n}|x| \longrightarrow 2|x|$$

as $n \to \infty$

Hence

$$\sum_{n=1}^{\infty} a_n \quad \text{conv} \quad \text{if} \quad 2|x| < 1 \quad \text{, ie} \quad |x| < \frac{1}{2}$$

$$\sum_{n=1}^{\infty} a_n \quad \text{div.} \quad \text{if} \quad 2|x| > 1 \quad \text{, ie} \quad |x| > \frac{1}{2}$$

$$\Rightarrow \quad R = \frac{1}{2}$$

$$x = \frac{-1}{2} \quad \Rightarrow \quad \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n n^2$$

$$(-1)^n n^2 \not\to 0 \quad \text{as} \quad n \to \infty \quad \Rightarrow \quad \text{divergent}$$

$$x = \frac{1}{2} \quad \Rightarrow \quad \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} n^2$$

$$n^2 \not\to 0 \quad \text{as} \quad n \to \infty \quad \Rightarrow \quad \text{divergent}$$

$$\Rightarrow \quad \text{I.o.c.} = \left(\frac{-1}{2}, \frac{1}{2} \right)$$

Q20/ $\quad a_n = \dfrac{(2x-1)^n}{5^n \sqrt{n}}$

$$\Rightarrow \left| \frac{a_{n+1}}{a_n} \right| = \frac{|2x-1|}{5} \cdot \frac{\sqrt{n}}{\sqrt{n+1}}$$

$$= \frac{|2x-1|}{5} \cdot \sqrt{\frac{n}{n+1}} \to \frac{|2x-1|}{5}$$

as $n \to \infty$

Hence

$\sum_{n=1}^{\infty} a_n$ conv if $\frac{|2x-1|}{5} < 1 \iff |2x-1| < 5$

$\iff 2\left|x - \frac{1}{2}\right| < 5 \iff \left|x - \frac{1}{2}\right| < \frac{5}{2}$

$\sum_{n=1}^{\infty} a_n$ div if $\frac{|2x-1|}{5} < 1 \iff \left|x - \frac{1}{2}\right| > \frac{5}{2}$

$\Rightarrow a = \frac{1}{2}$ and $R = \frac{5}{2}$

$x = -2 \Rightarrow \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ conv.

by A.S.T.

$x = 3 \Rightarrow \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ div.

(p-series with $p = \frac{1}{2} < 1$)

$\Rightarrow$ I.O.C. $= [-2, 3)$

Q24/ $a_n = \dfrac{n^2 x^n}{2 \cdot 4 \cdots 2n}$

$$\Rightarrow \left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|}{2(n+1)} \cdot \frac{(n+1)^2}{n^2}$$

$$= \frac{|x|}{2} \cdot \frac{n+1}{n^2} = \frac{|x|}{2} \frac{1/n + 1/n^2}{1} \to 0$$

as $n \to \infty$

$\Rightarrow \sum_{n=1}^{\infty} a_n$ convergent for all x

$\Rightarrow R = \infty$ and I.O.C. $= (-\infty, \infty)$.

Q30/ $\sum_{n=0}^{\infty} c_n x^n$ convergent at $x = -4$

$\Rightarrow R \geq 4$

$\sum_{n=0}^{\infty} c_n x^n$ divergent at $x = 6$

$\Rightarrow R \leq 6$

a) $\sum_{n=0}^{\infty} c_n = \sum_{n=0}^{\infty} c_n \cdot 1^n \qquad 1 < 4 \leq R$

$\Rightarrow$ convergent

b) $8 > 6 \geqslant R$

$$\Rightarrow \sum_{n=0}^{\infty} c_n 8^n \quad \text{divergent}$$

c) $|-3| < 4$ and $R \geqslant 4 \Rightarrow$

$$\sum_{n=0}^{\infty} c_n (-3)^n \quad \text{convergent}$$

d) $$\sum_{n=0}^{\infty} (-1)^n c_n 9^n = \sum_{n=0}^{\infty} c_n (-9)^n$$

$$|-9| = 9 > 6 \geqslant R \Rightarrow$$

$$\sum_{n=0}^{\infty} (-1)^n c_n 9^n \quad \text{divergent.}$$

Q37/ $1 + 2x + x^2 + 2x^3 + x^4 + \dots$

$$= 1 + x + x^2 + x^3 + x^4 \dots$$

$$+ \quad x \quad + \quad x^3 \quad + \quad x^5 \dots$$

$$1 + x + x^2 + x^3 + \dots = \frac{1}{1-x} \quad \text{if } |x| < 1$$

$$x + x^3 + x^5 + \dots = x(1 + x^2 + x^4 + \dots)$$

$$= \frac{x}{1-x^2} \quad \text{if } |x| < 1$$

For $x > 1$ both

$1 + x + x^2 + \dots$ and $x + x^3 + x^5 + \dots$

are divergent sums of positive terms

$\Rightarrow$ $1 + 2x + x^2 + 2x^3 + \dots$ divergent for $x > 1$

$\Rightarrow$ $1 + 2x + x^2 + 2x^3 + \dots$ convergent for $|x| < 1$ and divergent for $|x| > 1$

$\Rightarrow$ $R = 1$

For $x = 1$ we have the sum

$1 + 2 + 1 + 2 \dots$ which is divergent as the terms do <u>not</u> tend to zero.

For $x = -1$ we have the sum

$1 - 2 + 1 - 2 + 1 \ldots$ which is again

divergent as the terms do not tend to zero.

$\Rightarrow$ I.O.C. $= (-1, 1)$

and for $|x| < 1$ we have

$$1 + 2x + x^2 + 2x^3 + \ldots = \frac{1}{1-x} + \frac{x}{1-x^2}$$

Q4/ The sum of 2 convergent series is convergent. The sum of a convergent and divergent series is divergent.

$$\sum_{n=0}^{\infty} c_n x^n \quad , \quad R = 2$$

$$\sum_{n=0}^{\infty} d_n x^n \quad , \quad R = 3$$

$2 < 3 \quad \Rightarrow$

$$\sum_{n=0}^{\infty} (c_n + d_n) x^n \quad \text{conv if} \quad |x| < 2$$

and divergent if $|x| > 2 \quad \Rightarrow R = 2.$

§11.9 Functions as Power Series

Q3/ $\dfrac{1}{1-x} = 1 + x + x^2 + \dots \iff |x| < 1$

$\Rightarrow \dfrac{1}{1+x} = 1 - x + x^2 - x^3 \iff |x| < 1$

$= \displaystyle\sum_{n=0}^{\infty} (-1)^n x^n$

I.O.C. $= (-1, 1)$

Q6/ $f(x) = \dfrac{4}{2x+3} = \dfrac{4}{3\left(1 + \frac{2x}{3}\right)}$

$= \dfrac{4}{3} \cdot \dfrac{1}{1 - \left(\frac{-2x}{3}\right)}$

$= \dfrac{4}{3}\left(1 + \left(\dfrac{-2x}{3}\right) + \left(\dfrac{-2x}{3}\right)^2 + \dots\right)$

$= \displaystyle\sum_{n=0}^{\infty} \frac{4}{3}\left(-\frac{2x}{3}\right)^n = \sum_{n=0}^{\infty} (-1)^n \frac{2^{n+2}}{3^{n+1}} x^n$

$\iff \left|\dfrac{-2x}{3}\right| < 1 \iff |x| < \dfrac{3}{2}$

$\Rightarrow$ I.O.C. $\left(-\dfrac{3}{2}, \dfrac{3}{2}\right)$

Q7/ $f(x) = \dfrac{x^2}{x^4+16}$

$$= \frac{x^2}{16} \cdot \frac{1}{1+\left(\frac{x}{2}\right)^4} = \frac{x^2}{16} \cdot \frac{1}{1-\left(-\left(\frac{x}{2}\right)^4\right)}$$

$$= \frac{x^2}{16} \sum_{n=0}^{\infty} \left(-\left(\frac{x}{2}\right)^4\right)^n \qquad \left(\text{if } \left|-\frac{x^4}{16}\right| < 1\right)$$

$$\Updownarrow$$

$$|x| < 2$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{2^{4n+4}}$$

I.O.C. $= (-2, 2)$

Q13/a) $\dfrac{1}{1+x} = 1 - x + x^2 - x^3 \ldots \quad |x| < 1$

$$\Rightarrow \quad \frac{-1}{(x+1)^2} = -1 + 2x - 3x^2 + 4x^3 \ldots \quad |x| < 1$$

$$\Rightarrow \quad \frac{1}{(1+x)^2} = 1 - 2x + 3x^2 - 4x^3 \ldots \quad |x| < 1$$

$$= \sum_{n=0}^{\infty} (-1)^n (n+1) x^n \qquad |x| < 1$$

$R = 1$

b) $$\frac{-1}{(1+x)^2} = -1 + 2x - 3x^2 + 4x^3 \ldots \quad |x| < 1$$

$$\Rightarrow \frac{2}{(1+x)^3} = 2 - 3.2x + 4.3.x^2 \ldots \quad |x| < 1$$

$$\Rightarrow \frac{1}{(1+x)^3} = \frac{1}{2} \sum_{n=0}^{\infty} (-1)^n (n+2)(n+1) x^n$$

$|x| < 1$, $R = 1$

c) $$\frac{x^2}{(1+x)^3} = \sum_{n=0}^{\infty} \frac{(-1)^n (n+2)(n+1)}{2} x^{n+2}$$

$$= \sum_{n=2}^{\infty} \frac{(-1)^n n(n-1) x^n}{2}$$

$|x| < 1$, $R = 1$.

Q16/ In tecture we showed that

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} \ldots \quad \text{for } |x| < 1$$

$\Rightarrow x^2 \arctan(x^3)$

$$= x^2 \left(x^3 + \frac{(x^3)^3}{3} + \frac{(x^3)^5}{5} \ldots \right) \quad |x| < 1$$

$$= x^5 - \frac{x^{11}}{3} + \frac{x^{17}}{5} \ldots \qquad |x| < 1$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+5}}{2n+1} \qquad |x| < 1$$

$$a_n = \frac{x^{6n+5}}{2n+1} \Rightarrow \left|\frac{a_{n+1}}{a_n}\right| = \frac{2n+1}{2n+3} \cdot |x|^6$$

$$\longrightarrow |x|^6 \quad \text{as} \quad n \to \infty$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n \quad \text{conv} \quad \text{if} \quad |x|^6 < 1 \Leftrightarrow |x| < 1$$

$$\sum_{n=0}^{\infty} a_n \quad \text{div} \quad \text{if} \quad |x|^6 > 1 \Leftrightarrow |x| > 1$$

$$\Rightarrow R = 1$$

Q28/ $$\frac{\arctan(x)}{x} = \frac{1}{x}\left(x - \frac{x^3}{3} + \frac{x^5}{5} \ldots\right)$$

$$= 1 - \frac{x^2}{3} + \frac{x^4}{5} - \frac{x^6}{7} \ldots \qquad |x| < 1$$

$$\Rightarrow \int \frac{\arctan(x)}{x}\, dx = C + x - \frac{x^3}{3^2} + \frac{x^5}{5^2} \ldots$$

for $|x| < 1$

$$\Rightarrow \int \frac{\arctan(x)}{x}\,dx$$

$$= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)^2} \quad \text{if } |x|<1$$

$$a_n = (-1)^n \frac{x^{2n+1}}{(2n+1)^2} \Rightarrow \left|\frac{a_{n+1}}{a_n}\right| = |x|^2 \frac{(2n+1)^2}{(2n+3)^2}$$

$$\longrightarrow |x|^2 \quad \text{as} \quad n \to \infty$$

$$|x|^2 < 1 \iff |x| < 1$$

$$|x|^2 > 1 \iff |x| > 1$$

$$\Rightarrow R = 1$$

Q34/ Firstly if we do the ratio test we find $R = \infty \Rightarrow$

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$ makes sence for all x.

$$f'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2n x^{2n-1}}{(2n)!} = \frac{-2x}{2!} + \frac{4x^3}{4!} - \frac{6x^5}{6!} \cdots$$

$$= -x + \frac{x^3}{3!} - \frac{x^5}{5!} \dots$$

$$\Rightarrow f''(x) = -1 + \frac{x^2}{2!} - \frac{x^4}{4!} \dots$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n}}{(2n)!}$$

$$\Rightarrow f''(x) + f(x) = \sum_{n=0}^{\infty} \left((-1)^n + (-1)^{n+1}\right) \frac{x^{2n}}{(2n)!}$$

$$= 0$$

Q41/ $\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} \dots \qquad |x| < 1$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$\tan\left(\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \arctan\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6} \qquad \left(\left|\frac{1}{\sqrt{3}}\right| < 1\right)$$

$$= \frac{\pi}{6} = \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{1}{\sqrt{3}}\right)^{2n+1}}{2n+1}$$

$$\Rightarrow \frac{\pi}{6} = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{\sqrt{3}} \cdot \frac{1}{(2n+1)\, 3^n}$$

$$\Rightarrow \pi = \frac{6}{\sqrt{3}} \sum_{n=0}^{\infty} (-1)^n \cdot \frac{1}{(2n+1)\, 3^n}$$

$$= 2\sqrt{3} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)\, 3^n}.$$

§ 11.10 Taylor and Maclaurin Series

Q1/ $b_8 = \dfrac{f^{(8)}(5)}{8!}$

Q3/ $f^{(n)}(0) = (n+1)! \Rightarrow c_n = \dfrac{(n+1)!}{n!} = n+1$

$\Rightarrow$ Maclaurin Series $= 1 + 2x + 3x^2 + 4x^3 + \ldots$

$= \displaystyle\sum_{n=0}^{\infty} (n+1) x^n$

$a_n = (n+1)x^n \Rightarrow \left| \dfrac{a_{n+1}}{a_n} \right| = \dfrac{n+1}{n} |x| \to |x|$

as $n \to \infty$

Maclaurin series conv. if $|x| < 1$

Maclaurin Series div. if $|x| > 1$

$\Rightarrow R = 1$

Q4/ $f^{(n)}(4) = \dfrac{(-1)^n n!}{3^n (n+1)} \Rightarrow c_n = \dfrac{(-1)^n}{3^n (n+1)}$

$\Rightarrow$ Taylor series at 4 $= \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{3^n (n+1)} (x-4)^n$

$$a_n = \frac{(-1)^n}{3^n(n+1)}(x-4)^n$$

$$\Rightarrow \left|\frac{a_{n+1}}{a_n}\right| = \frac{n}{n+1}\cdot\frac{1}{3}\cdot|x-4|$$

$$\Rightarrow \left|\frac{a_{n+1}}{a_n}\right| \to \frac{1}{3}\cdot|x-4| \quad \text{as} \quad n\to\infty$$

Convergent if $\frac{1}{3}|x-4| < 1 \Leftrightarrow |x-4| < 3$

Divergent if $\frac{1}{3}|x-4| > 1 \Leftrightarrow |x-4| > 3$

$\Rightarrow \quad R = 3.$

Q8/ $f(x) = \ln(x) \Rightarrow f'(x) = \frac{1}{x}$

$\Rightarrow f''(x) = \frac{-1}{x^2} \Rightarrow f'''(x) = \frac{2}{x^3}$

$\Rightarrow f^{(4)}(x) = \frac{-6}{x^4}$

$\Rightarrow f(1) = 0, \quad f'(1) = 1, \quad f''(1) = -1,$

$f'''(1) = 2, \quad f^{(4)}(1) = -6$

Taylor series at $x=1$ $= (x-1) - \frac{(x-1)^2}{2!} + \frac{2(x-1)^3}{3!}$

$- \frac{6(x-1)^4}{4!} + \ldots$

Q14/ $f(x) = x^5 + 2x^3 + x \quad , a = 2$

$f'(x) = 5x^4 + 6x^2 + 1$

$f''(x) = 20x^3 + 12x$

$f'''(x) = 60x^2 + 12$

$f^{(4)}(x) = 120x$

$f^{(5)}(x) = 120$

$\vdots$

$f^{(n)}(x) = 0$

$\Rightarrow \quad f(2) = 32 + 16 + 2 = 50$

$f'(2) = 80 + 24 + 1 = 105$

$f''(2) = 160 + 24 = 184$

$f'''(2) = 240 + 12 = 252$

$f^{(4)}(2) = 240$

$f^{(5)}(2) = 120$

$\vdots$

$f^{(n)}(2) = 0$

$\Rightarrow$ Taylor series at $x = 1$

$$= 50 + 105(x-2) + 92(x-2)^2$$

$(R = \infty)$ $$+ 42(x-2)^3 + 10(x-2)^4 + (x-2)^5$$

Q25/ $f(x) = \sin(x)$ $\quad f(\pi) = 0$

$f'(x) = \cos(x)$ $\quad f'(\pi) = -1$

$f''(x) = -\sin(x)$ $\Rightarrow$ $f''(\pi) = 0$

$f'''(x) = -\cos(x)$ $\quad f'''(\pi) = 1$

$f^{(4)}(x) = \sin(x)$ $\quad f^{(4)}(\pi) = 0$

Taylor series of $\sin(x)$ at $x = \pi$

$$= -(x-\pi) + \frac{(x-\pi)^3}{3!} - \frac{(x-\pi)^5}{5!} \cdots$$

$$= \sum_{n=0}^{\infty} (-1)^{n+1} \frac{(x-\pi)^{2n+1}}{(2n+1)!}$$

$$a_n = \frac{(-1)^{n+1}(x-\pi)^{2n+1}}{(2n+1)!} \Rightarrow \left|\frac{a_{n+1}}{a_n}\right| = \frac{|x-\pi|^2}{(2n+2)(2n+3)}$$

$\longrightarrow 0$ as $n \to \infty$

$\Rightarrow R = \infty$ by Ratio Test.

Q 28

Claim

$$\underset{\text{}}{\overset{f(x)}{\overset{\shortparallel}{\sin(x)}}} = -(x-\pi) + \frac{(x-\pi)^3}{3!} - \frac{(x-5)^5}{5!} + \dots$$

for all x.

Proof

For $n \geqslant 0$ $\quad f^{(n+1)}(x) = \pm \sin(x)$

or $\pm \cos(x) \quad \Rightarrow \quad |f^{(n+1)}(x)| \leqslant 1$

for all x in $\mathbb{R}$

$$\Rightarrow \quad |R_n(x)| \leqslant \frac{|x-\pi|^{n+1}}{(n+1)!} \quad \text{for all } n \geqslant 0$$

$$\sum_{n=0}^{\infty} \frac{|x-\pi|^{n+1}}{n!} \text{ conv} \quad \Rightarrow \quad \lim_{n \to \infty} \left\{ \frac{|x-\pi|^{n+1}}{n!} \right\} = 0$$

$$\Rightarrow \quad \lim_{n \to \infty} \left\{ \frac{|x-\pi|^{n+1}}{(n+1)!} \right\} = 0 \quad \Rightarrow$$

$$\lim_{n \to \infty} |R_n(x)| = 0 \quad \Rightarrow$$

$$\sin(x) = -(x-\pi) + \frac{(x-\pi)^3}{3!} - \frac{(x-5)^5}{5!} + \dots$$

for all x in $\mathbb{R}$

Q 31/ $$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^k$$

if $|x| < 1$

Radius of convergence $= 1$

$$\Rightarrow \sqrt[4]{1-x} = (1+(-x))^{1/4}$$

$$= \sum_{n=0}^{\infty} \binom{1/4}{n} (-x)^n$$

if $|-x| < 1$ $\Updownarrow$ $|x| < 1$

$$= \sum_{n=0}^{\infty} (-1)^n \binom{1/4}{n} x^n$$

$$= 1 + \left(-\tfrac{1}{4}\right)x + \frac{\frac{1}{4}\cdot\left(\frac{1}{4}-1\right)}{2!} x^2$$

$$- \frac{\frac{1}{4}\cdot\left(\frac{1}{4}-1\right)\cdot\left(\frac{1}{4}-2\right)}{3!} x^3 + \ldots$$

$$= 1 - \frac{1}{4}x - \frac{3}{32}x^2 - \frac{1.3.7}{4^3.3!}x^3$$

$\ldots$

Radius of convergence $R = 1$.

Q41/

$$f(x) = \frac{x}{\sqrt{4+x^2}} = x\left(4+x^2\right)^{-\frac{1}{2}}$$

$$= \frac{1}{2}x\left(1+\left(\frac{x}{2}\right)^2\right)^{-\frac{1}{2}}$$

$$= \frac{1}{2}x\sum_{n=0}^{\infty}\binom{-1/2}{n}\left(\frac{x}{2}\right)^{2n}$$

$$= \sum_{n=0}^{\infty}\binom{-1/2}{n}\frac{x^{2n+1}}{2^{2n+1}}$$

$$= \frac{x}{2} + \frac{(-1/2)}{1!}\frac{x^3}{2^3} + \frac{(-1/2)(-3/2)}{2!}\cdot\frac{x^5}{2^5}$$

$$+ \frac{(-1/2)(-3/2)(-5/2)}{3!}\cdot\frac{x^7}{2^7} + \dots$$

$$\dots + \frac{(-1)^n(1.3.5\dots(2n-1))}{n!}\cdot\frac{x^{2n+1}}{2^{3n+1}} + \dots$$

If $\left|\left(\frac{x}{2}\right)^2\right| < 1 \iff |x| < 2$

Q55/

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \dots \quad \text{for all } x$$

$$\Rightarrow \frac{\cos(x) - 1}{x} = \frac{-x}{2!} + \frac{x^3}{4!} - \frac{x^5}{6!} \dots$$

$$\Rightarrow \int \frac{\cos(x) - 1}{x}\, dx = \frac{-x^2}{2.(2!)} + \frac{x^4}{4.(4!)} - \frac{x^6}{6.(6!)} + \dots + C$$

$$= C + \sum_{n=0}^{\infty} \frac{(-1)^{n-1} x^{2n+2}}{(2n+2)\cdot((2n+2)!)}.$$

Q75/

Recall that $\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$

for $|x| < 1$

$\left|\frac{3}{5}\right| < 1 \Rightarrow \ln\left(1 + \frac{3}{5}\right) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{3^n}{n\, 5^n}$

Q79/ $3 + \frac{9}{2!} + \frac{27}{3!} + \frac{81}{4!} + \dots + \frac{3^n}{n!} + \dots$

$$e^3 = 1 + 3 + \frac{3^2}{2!} + \frac{3^3}{3!} + \dots$$

$$\Rightarrow 3 + \frac{9}{2!} + \frac{27}{3!} + \frac{81}{4!} + \dots = e^3 - 1$$

Q 84 a) /

$$f(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

<u>Claim</u> $f^{(n)}(0) = 0$ for all $n \geq 0$

Assume $x \neq 0 \Rightarrow$

$$f'(x) = \frac{2}{x^3} e^{-1/x^2}$$

$$f''(x) = \frac{-6}{x^4} e^{-1/x^2} + \frac{4}{x^6} e^{-1/x^2}$$

$$f'''(x) = \frac{24}{x^5} e^{-1/x^2} - \frac{12}{x^7} e^{-1/x^2} - \frac{24}{x^7} e^{-1/x^2}$$

$$+ \frac{8}{x^9} e^{-1/x^2}$$

If we continue we see that $f^{(n)}(x)$ will be a finite sum of terms of the form $\frac{A}{x^k} e^{-1/x^2}$.

where A is constant k is a positive integer. Consider $\lim_{x \to 0} \frac{A}{x^k} e^{-1/x^2}$.

As $x \to 0$, $1/x^2 \to \infty$

$$\Rightarrow e^{-1/x^2} \to 0$$

We wish to show that

$$\lim_{x \to 0} \frac{A}{x^k} e^{-1/x^2} = 0$$

To do this it will suffice to show

$$\lim_{x \to 0} \frac{|A|}{|x|^k} e^{-1/x^2} = 0$$

Let $u = 1/x^2 \Rightarrow |x| = \sqrt{\frac{1}{u}}$

As $x \to 0$, $u \to \infty$. Hence

$$\lim_{x \to 0} \frac{|A|}{|x|^k} e^{-1/x^2} = \lim_{u \to \infty} \frac{|A|}{\left(\frac{1}{u}\right)^{k/2}} e^{-u}$$

$$= \lim_{u \to \infty} \frac{|A| \, u^{k/2}}{e^u}$$

If we now apply L'Hopitals rule repeatedly we eventually have the numerator with u to power $d \le 0$. Hence the numerator tends to 0 and the denominator (e^u) to ∞.

Hence the limit is 0.

$$\Rightarrow \lim_{u \to \infty} \frac{|A| u^{k/2}}{e^u} = 0$$

$$\Rightarrow \lim_{x \to 0} \left| \frac{A}{x^k} e^{-1/x^2} \right| = 0$$

$$\Rightarrow \lim_{x \to 0} \frac{A}{x^k} e^{-1/x^2} = 0$$

$f^{(n)}(x)$ is the sum of such expressions

$$\Rightarrow \lim_{x \to 0} f^{(n)}(x) = 0$$

$\Rightarrow f^{(n)}(0) = 0$ for all $n \geq 0$

$\Rightarrow$ Taylor series at $x = 0$ is just 0.

However $f(x)$ is <u>not</u> the zero function on any open interval containing 0 ($e^{-1/x^2} \neq 0$ if $x \neq 0$). Hence $f(x)$ is <u>not</u> equal to its Maclaurin series on any open interval containing 0.