Homework 6 Solutions

§11.5 Alternating Series

Q3/ $$-\frac{2}{5}+\frac{4}{6}-\frac{6}{7}+\frac{8}{8}\dots = \sum_{n=1}^{\infty} a_n$$

$$a_n = (-1)^n \frac{2n}{n+4}$$

$$\frac{2n}{n+4} \to 2 \quad \text{as} \quad n \to \infty \quad \Rightarrow \lim_{n\to\infty} a_n \neq 0$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n \quad \text{div.}$$

Q4/ $$a_n = (-1)^{n-1} \frac{1}{\ln(n+2)}$$

$$b_n = \frac{1}{\ln(n+2)}, \quad f(x) = \frac{1}{\ln(x+2)}$$

$$f'(x) = \frac{\frac{-1}{x+2}}{\ln(x+2)^2} < 0 \quad \text{for} \quad x \geq 1$$

$\Rightarrow$ $\{b_n\}$ decreasing

$\ln(x+2) \to \infty$ as $x \to \infty$ $\Rightarrow$ $b_n \to 0$ as $n \to \infty$

$\Rightarrow$ convergent by A.S.T.

Q8/ $b_n = \dfrac{n^2}{n^2+n+1} = \dfrac{1}{1+\frac{1}{n}+\frac{1}{n^2}}$

$\longrightarrow 1$ as $n \to \infty$

$\Rightarrow \lim\limits_{n\to\infty} (-1)^n \dfrac{n^2}{n^2+n+1} \neq 0$

$\Rightarrow$ divergent.

Q12/ $b_n = \dfrac{n}{e^n}$, $f(x) = \dfrac{x}{e^x}$

$\Rightarrow f'(x) = \dfrac{e^x - x e^x}{e^{2x}} = \dfrac{1-x}{e^x} \leq 0$

for $x \geqslant 1$

$\Rightarrow \{b_n\}$ decreasing

$\lim\limits_{x\to\infty} \dfrac{x}{e^x} \underset{\uparrow}{=} \lim\limits_{x\to\infty} \dfrac{1}{e^x} = 0$

L'Hospitale

$\Rightarrow$ convergent by A.S.T.

Q16/ $\cos(n\pi) = (-1)^n \Rightarrow$

$$\sum_{n=1}^{\infty} \frac{n\cos(n\pi)}{2^n} = \sum_{n=1}^{\infty} (-1)^n \frac{n}{2^n}$$

$$b_n = \frac{n}{2^n}, \quad f(x) = \frac{x}{2^x}$$

$$f'(x) = \frac{2^x - x\ln(2)\,2^x}{2^{2x}} = \frac{(1 - x\ln(2))}{2^x} < 0$$

for $x \geq 2$ $\quad (\ln(2) > 0.5)$

$\Rightarrow \{b_n\}$ decreasing for $n \geq 2$

$$\lim_{x \to \infty} \frac{x}{2^x} = \lim_{x \to \infty} \frac{1}{\ln(2)\,2^x} = 0$$

$$\Rightarrow \lim_{n \to \infty} \{b_n\} = 0$$

$\Rightarrow$ convergent by A.S.T.

Q19/ $$\frac{n^n}{n!} = \frac{n \times n \times n \ldots \times n}{n \times (n-1) \times (n-2) \ldots \times 1} \geq 1$$

for $n > 1$

$$\Rightarrow \lim_{n \to \infty} (-1)^n \frac{n^n}{n!} \neq 0 \Rightarrow \text{div.}$$

Q34/

$$b_n = \frac{\ln(n)^p}{n}, \quad f(x) = \frac{\ln(x)^p}{x}$$

$$f'(x) = \frac{(p - \ln(x))}{x^2} \ln(x)^{p-1} \Rightarrow$$

$f'(x) < 0$ for all $x > e^p$

$\Rightarrow$ $\{b_n\}$ eventually decreasing.

$$\lim_{x \to \infty} \ln(x) = \infty$$

If $p \leq 0 \Rightarrow \frac{\ln(x)^p}{x} \to 0$

as $x \to \infty$

Assume $p > 0$

$$\lim_{x \to \infty} \frac{\ln(x)^p}{x} = \lim_{x \to \infty} \frac{\frac{p \ln(x)^{p-1}}{x}}{1}$$

$$= \lim_{x \to \infty} \frac{p \ln(x)^{p-1}}{x}$$

If p is an integer we repeat this

p times to get

$$\lim_{x \to \infty} \frac{\ln(x)^p}{x} = \lim_{x \to \infty} \frac{p!}{x} = 0$$

If p is not an integer then after repeating L'Hopital enough we end up in the $p < 0$ case.

$$\Rightarrow \lim_{x \to \infty} \frac{\ln(x)^p}{x} = 0 \quad \text{for all } p.$$

$$\Rightarrow \lim_{n \to \infty} \left\{ \frac{\ln(n)^p}{n} \right\} = 0 \quad \text{for all } p$$

$$\Rightarrow \sum_{n=2}^{\infty} (-1)^{n-1} \frac{\ln(n)^p}{n} \quad \text{conv. for all } p.$$

§11.6 Absolute Convergence

Q2/ $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$. Div $(p = \frac{1}{2} < 1)$

$\Rightarrow$ <u>not</u> abs. conv.

$\left\{\frac{1}{\sqrt{n}}\right\}$ positive, decreasing and $\lim_{n \to \infty} \left\{\frac{1}{\sqrt{n}}\right\} = 0$

$\Rightarrow \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ conv by A.S.T. $\Rightarrow$ <u>conditionally conv.</u>

Q5/ $a_n = \frac{\sin(n)}{2^n}$, $|a_n| = \frac{|\sin(n)|}{2^n}$

$\Rightarrow |a_n| < \left(\frac{1}{2}\right)^n$

$\sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ conv $\Rightarrow \sum_{n=1}^{\infty} \frac{|\sin(n)|}{2^n}$ conv.

$\Rightarrow$ Abs. Conv.

Q7/ $a_n = \frac{n}{5^n} \Rightarrow \left|\frac{a_{n+1}}{a_n}\right| = \frac{n+1}{n} \cdot \frac{1}{5}$

$\frac{n+1}{n} = \frac{1 + \frac{1}{n}}{1} \longrightarrow 1$ as $n \to \infty$

$\Rightarrow \lim_{n \to \infty} \left\{ \left|\frac{a_{n+1}}{a_n}\right| \right\} = \frac{1}{5} < 1$

$\Rightarrow$ Abs. conv. by ratio test.

Q12/ $a_k = ke^{-k}$

$\Rightarrow \left|\frac{a_{k+1}}{a_k}\right| = \frac{k+1}{k} \cdot \frac{1}{e} \to \frac{1}{e} < 1$

as $k \to \infty$

$\Rightarrow$ convergent.

Q20/ $a_n = \frac{(2n)!}{(n!)^2} \Rightarrow \left|\frac{a_{n+1}}{a_n}\right| = \frac{(2n+2)(2n+1)}{(n+1)(n+1)}$

$\Rightarrow \left|\frac{a_{n+1}}{a_n}\right| \to 4 > 1$ as $n \to \infty$

$\Rightarrow$ divergent.

Q23/ $a_n = \frac{2 \cdot 4 \cdot 6 \cdot \ldots \cdot 2n}{n!}$

$\left|\frac{a_{n+1}}{a_n}\right| = \frac{2n+2}{n+1} \to 2 > 1$ as $n \to \infty$

$\Rightarrow$ divergent

Q31/ $\ln(n) \to \infty$ as $n \to \infty \Rightarrow$

$\frac{1}{\ln(n)} \to 0$ as $n \to \infty$

$$n+1 > n > 2 \Rightarrow \ln(n+1) > \ln(n) > 0$$

$$\Rightarrow \frac{1}{\ln(n)} > \frac{1}{\ln(n+1)} \Rightarrow \left\{ \frac{1}{\ln(n)} \right\}$$

decreasing

$$\Rightarrow \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)} \quad \text{conv. by A.S.T.}$$

Q42/ $a_n = \dfrac{(-1)^n n!}{n^n b_1 \ldots b_n}$

$$\Rightarrow \left| \frac{a_{n+1}}{a_n} \right| = \frac{n+1}{|b_{n+1}|} \cdot \frac{n^n}{(n+1)^{n+1}}$$

$$= \frac{1}{|b_{n+1}|} \cdot \left(\frac{n}{n+1} \right)^n$$

$$\left(\frac{n+1}{n} \right)^n = \left(1 + \frac{1}{n} \right)^n \longrightarrow e \quad \text{as } n \to \infty$$

$$\Rightarrow \left| \frac{a_{n+1}}{a_n} \right| \longrightarrow \frac{1}{\frac{1}{2}} \cdot \frac{1}{e} = \frac{2}{e} < 1$$

$\Rightarrow$ conv. by ratio test.

Q 44/ $a_n = \dfrac{(n!)^2}{(kn)!}$

$$\Rightarrow \left| \frac{a_{n+1}}{a_n} \right| = \frac{(n+1)^2}{(kn+1)\cdots(kn+k)}$$

$$k = 1 \Rightarrow \left| \frac{a_{n+1}}{a_n} \right| = n+1 \to \infty$$

$$k = 2 \Rightarrow \left| \frac{a_{n+1}}{a_n} \right| \to \frac{1}{4}$$

$$k > 2 \Rightarrow \left| \frac{a_{n+1}}{a_n} \right| \to 0$$

$\Rightarrow$ Convergent if and only if $k \geq 2$.

Optional Problems

Q 51/ a) $S_n^{|\cdot|} = |a_1| + \dots + |a_n|$

$$S_n^+ = a_1^+ + \dots + a_n^+$$

For all $k \geq 1$ $\quad a_k^+ \leq |a_k|$

$\Rightarrow 0 \leq S_n^+ \leq S_n^{|\cdot|}$ for all n.

$\{S_n^{|\cdot|}\}$ increasing and by definition

$$\lim_{n\to\infty} \{S_n^{1.1}\} = S = \sum_{n=1}^{\infty} |a_n|$$

$\Rightarrow \quad S_n^{1.1} \leq S$ for all n.

$a_k^+ \geqslant 0$ for all $k \geqslant 1 \quad \Rightarrow$

$\{S_n^+\}$ increasing and bounded above

$\Rightarrow$ bounded and monotone

$\Rightarrow$ convergent

$\Rightarrow \sum_{n=1}^{\infty} a_n^+$ convergent by definition.

$$|a_n| = a_n^+ - a_n^-$$

$$\Rightarrow \quad a_n^- = a_n^+ - |a_n|$$

$\sum_{n=1}^{\infty} |a_n|, \ \sum_{n=1}^{\infty} a_n^+$ conv.

$\Rightarrow \sum_{n=1}^{\infty} (a_n^+ - |a_n|)$ conv.

$\Rightarrow \sum_{n=1}^{\infty} a_n^-$ conv.

Hence $\sum_{n=1}^{\infty} |a_n|$ conv $\Rightarrow \sum_{n=1}^{\infty} a_n^+, \sum_{n=1}^{\infty} a_n^-$ conv.

b) $\sum_{n=1}^{\infty} a_n$ cond. conv.

$\Rightarrow \sum_{n=1}^{\infty} a_n$ div and $\sum_{n=1}^{\infty} |a_n|$ div.

$|a_n| = a_n^+ - a_n^-$

$a_n = a_n^+ + a_n^-$

Assume $\sum_{n=1}^{\infty} a_n$ is conditionally convergent.

Case 1 $\sum a_n^+$ conv and $\sum a_n^-$ conv.

$\Rightarrow \sum_{n=1}^{\infty} (a_n^+ - a_n^-)$ conv

$\Rightarrow \sum_{n=1}^{\infty} |a_n|$ conv $\Rightarrow$ Not cond. conv.

Case 2 $\sum_{n=1}^{\infty} a_n^+$ conv, $\sum_{n=1}^{\infty} a_n^-$ div

$a_n = a_n^+ + a_n^- \Rightarrow a_n^- = a_n - a_n^+$

$\sum_{n=1}^{\infty} a_n$ conv, $\sum_{n=1}^{\infty} a_n^+$ conv $\Rightarrow$

$\sum_{n=1}^{\infty} (a_n - a_n^+) = \sum_{n=1}^{\infty} a_n^-$ conv.

Contradiction.

Let s be a real number.

We want to rearrange $\{a_n\}$ to make the sum s.

We'll define $\{d_n\}$ via the following algorithm

1/ If $s \geqslant 0$ let $d_1 = b_1$ ← 1st + term

If $s < 0$ let $d_1 = c_1$ ← 1st − term

2/ Imagine we have defined $d_1, \dots, d_{k-1}$.

These will composed of b_i and c_j for varying i and j.

If $d_1 + \dots + d_{k-1} \geqslant s$ let d_k be the next unselected c_j.

If $d_1 + \dots + d_{k-1} < s$ let d_k be the next unselected b_i.

Example $\quad S = \frac{3}{4}$, $a_n = (-1)^{n-1}\frac{1}{n}$

$$\{b_n\} = \left\{1, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \dots\right\}$$

$$\{c_n\} = \left\{-\frac{1}{2}, -\frac{1}{4}, -\frac{1}{6}, -\frac{1}{8}, \dots\right\}$$

1/ $\frac{3}{4} > 0 \Rightarrow d_1 = b_1 = 1$

$$d_1 = 1 > \frac{3}{4} = S \Rightarrow d_2 = c_1 = -\frac{1}{2}$$

$$d_1 + d_2 = 1 - \frac{1}{2} = \frac{1}{2} < \frac{3}{4} = S$$

$$\Rightarrow d_3 = b_2 = \frac{1}{3}$$

$$d_1 + d_2 + d_3 = 1 - \frac{1}{2} + \frac{1}{3} = \frac{5}{6} > \frac{3}{4} = S$$

$$\Rightarrow d_4 = c_2 = -\frac{1}{4}$$

$$d_1 + d_2 + d_3 + d_4 = \frac{5}{6} - \frac{1}{4} = \frac{7}{12} < \frac{3}{4} = S$$

$\dots\dots$

Heuristically we add up enough b_i to be $\geqslant 5$, then add enough c_j to be < 5 and repeat.

This never stops because $\sum_{n=1}^{\infty} b_n$ and $\sum_{n=1}^{\infty} c_n$ are divergent by 5I b).

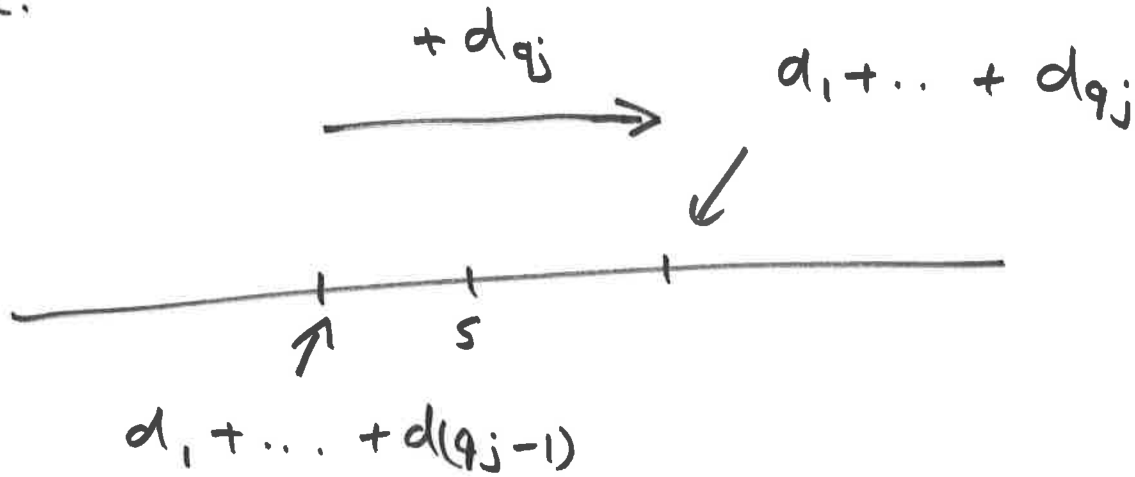
<u>Claim</u> $\sum_{n=1}^{\infty} d_n = 5$

<u>Proof</u> Let $\{q_1, q_2, \ldots\}$ be the sequence of whole numbers such that adding d_{q_j} (to the previous d_k's) crosses 5.

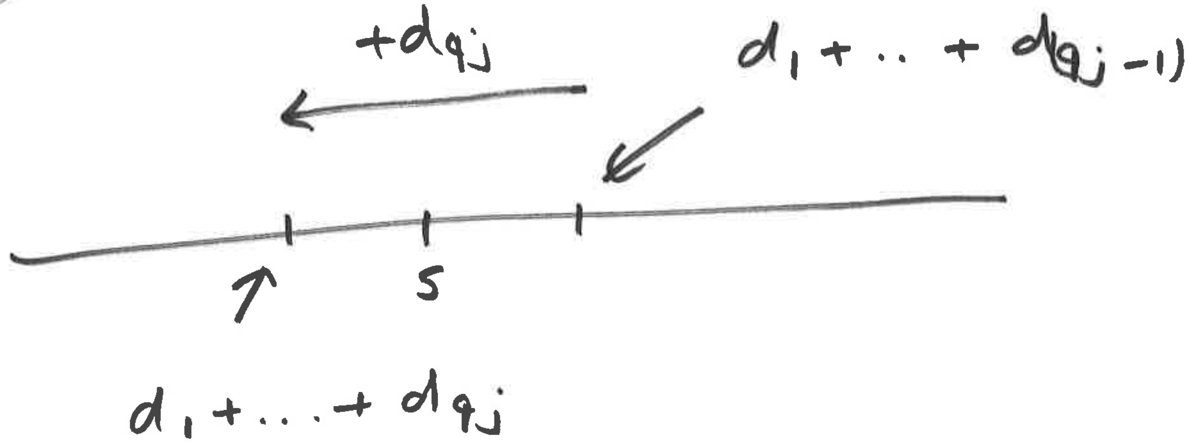
i.e. If $d_{q_j} \geqslant 0 \Rightarrow d_{q_j+1} < 0$

If $d_{q_j} < 0 \Rightarrow d_{q_j+1} \geqslant 0$

i.e.



or



$$\sum_{n=1}^{\infty} a_n \text{ conv} \Rightarrow \lim_{n \to \infty} \{a_n\} = 0$$

$$\Rightarrow \lim_{n \to \infty} \{d_n\} = 0 \Rightarrow \lim_{n \to \infty} \{d_{q_n}\} = 0$$

Hence given $\varepsilon > 0$ there is an $N > 0$ such that $|d_{q_n}| \leqslant \varepsilon$ for all $n \geqslant N$.

However observe that if

$S_k = d_1 + \dots + d_k$ then if

$k \geqslant q_N$ $\quad |s - s_k| \leqslant |d_{q_n}|$

for all $n \geqslant N$.

(This is the same logic as the error bound in the alternating series case)

$\Rightarrow \quad |s - s_k| \leq \varepsilon$ for all $n \geqslant q_N$

$$\Rightarrow \lim_{k \to \infty} \{S_k\} = s \Rightarrow \sum_{n=1}^{\infty} d_n = s.$$

Note that $\{d_1, d_2, \dots\}$ is a rearrangement of $\{a_1, a_2, a_3, \dots\}$.

Hence any sum s can be achieved with a suitable rearrangement.

§11.7 Strategies for Series Testing

Q3/ $$f(x) = \frac{x^2 - 1}{x^3 + 1}$$

$$\Rightarrow \quad f'(x) = \frac{2x(x^3+1) - (x^2-1)(3x^2)}{(x^3+1)^2}$$

$$= \frac{-x^3 + 3x^2 + 2x}{(x^3+1)^2}$$

$$= \frac{-x(x-3)(x+1)}{(x^3+1)^2}$$

$$\Rightarrow \quad f'(x) < 0 \quad \text{if} \quad x > 3$$

$$\Rightarrow \quad \left\{ \frac{n^2-1}{n^3+1} \right\} \text{ is eventually decreasing}$$

$$\frac{n^2-1}{n^3+1} = \frac{\frac{1}{n} - \frac{1}{n^3}}{1 + 1/n^3} \longrightarrow 0 \quad \text{as} \quad n \to \infty$$

$$\frac{n^2-1}{n^3+1} \geqslant 0 \quad \text{for all} \quad n \geq 1$$

$$\Rightarrow \sum_{n=1}^{\infty} (-1)^n \frac{n^2-1}{n^3+1} \quad \text{convergent by A.S.T.}$$

Q5/ $f(x) = \dfrac{e^x}{x^2}$

$$\lim_{x\to\infty} \frac{e^x}{x^2} = \lim_{x\to\infty} \frac{e^x}{2x} = \lim_{x\to\infty} \frac{e^x}{2}$$

$= \infty$

L'Hospitale

$$\Rightarrow \lim_{n\to\infty} \left\{ \frac{e^n}{n^2} \right\} \neq 0$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{e^n}{n^2} \quad \text{div.}$$

Q10/ $f(x) = x^2 e^{-x^3}$

$f(x) \geqslant 0$ for all $x \geq 1$

x^2, e^x, $-x^2$ continuous $\Rightarrow f(x)$ continuous on $[1, \infty)$.

$$f'(x) = 2x e^{-x^3} - 6x^3 e^{-x^3}$$

$$= 2x(1 - 3x^2) e^{-x^3}$$

$\Rightarrow f'(x) < 0$ if $x > \sqrt{\frac{1}{3}}$

$\Rightarrow f(x)$ decreasing on $[1, \infty)$

Let $u = -x^3 \Rightarrow \frac{du}{dx} = -3x^2$

$$\Rightarrow dx = \frac{du}{-3x^2}$$

$$\Rightarrow \int x^2 e^{-x^3} dx = \int \frac{-1}{3} e^u du$$

$$= \frac{-1}{3} e^u + C$$

$$= \frac{-1}{3} e^{-x^3} + C$$

$$\Rightarrow \int_1^\infty f(x)\,dx = \lim_{t\to\infty} \frac{-1}{3} e^{-x^3} \Big|_1^t$$

$$= \lim_{t\to\infty} \left(\frac{-1}{3} e^{-t^3} + \frac{1}{3} e^{-1} \right) = \frac{1}{3} e^{-1}$$

$\Rightarrow$ convergent

$\Rightarrow \displaystyle\sum_{n=1}^\infty n^2 e^{-n^3}$ convergent by Integral Test.

Q14/ $$0 \le \left| \frac{\sin 2n}{1+2^n} \right| \le \frac{1}{1+2^n} < \frac{1}{2^n}$$

$\displaystyle\sum_{n=1}^\infty \frac{1}{2^n} = \sum_{n=1}^\infty \left(\frac{1}{2}\right)^n$ convergent

as $\left|\frac{1}{2}\right| < 1 \Rightarrow \displaystyle\sum_{n=1}^\infty \left| \frac{\sin(2n)}{1+2^n} \right|$ conv.

$\Rightarrow \displaystyle\sum_{n=1}^\infty \frac{\sin(2n)}{1+2^n}$ conv.

Q17/ $$a_n = \frac{1 \cdot 3 \cdot 5 \; \ldots \; (2n-1)}{2 \cdot 5 \cdot 8 \; \ldots \; (3n-1)}$$

$$\Rightarrow \left| \frac{a_{n+1}}{a_n} \right| = \frac{2(n+1)-1}{3(n+1)-1} = \frac{2n+1}{3n+2}$$

$$= \frac{2 + 1/n}{3 + 2/n} \longrightarrow \frac{2}{3} < 1 \quad \text{as } n \to \infty$$

$$\Rightarrow \sum_{n=1}^{\infty} a_n \quad \text{convergent. by Ratio Test.}$$

Q23/ $a_n = \tan(1/n)$, $b_n = 1/n$

$$\frac{a_n}{b_n} = \frac{\tan(1/n)}{1/n}$$

$$f(x) = \frac{\tan(1/x)}{1/x} \quad , \quad \lim_{x \to \infty} \frac{\tan(1/x)}{1/x}$$

$$= \lim_{x \to \infty} \frac{\frac{-1}{x^2} \sec^2(1/x)}{\frac{-1}{x^2}} = \lim_{x \to \infty} \sec^2(1/x)$$

L'Hospitale

$$= \sec^2\left(\lim_{x\to\infty} 1/x\right) = \sec^2(0) = 1$$

$\uparrow$

$\sec^2$ continuous at $x = 0$

$$\Rightarrow \lim_{n\to\infty} \left\{\frac{a_n}{b_n}\right\} = 1$$

$$\Rightarrow \left(\sum_{n=1}^{\infty} \frac{1}{n} \text{ div.} \Rightarrow \sum_{n=1}^{\infty} \tan(1/n) \text{ div} \right)$$

by Limit comparison test.

Q25/ $\quad a_n = \dfrac{n!}{e^{n^2}}$

$$\Rightarrow \left|\frac{a_{n+1}}{a_n}\right| = (n+1) \cdot \frac{e^{n^2}}{e^{(n+1)^2}}$$

$$= (n+1)\, e^{n^2 - (n+1)^2}$$

$$= \frac{n+1}{e^{2n+1}}$$

$$f(x) = \frac{x+1}{e^{2x+1}}$$

$$\lim_{x \to \infty} \frac{x+1}{e^{2x+1}} \underset{\text{L'Hospitale}}{=} \lim_{x \to \infty} \frac{1}{2e^{2x+1}} = 0$$

$$\Rightarrow \lim_{n \to \infty} \left\{ \left| \frac{a_{n+1}}{a_n} \right| \right\} = 0 < 1$$

$$\Rightarrow \sum_{n=1}^{\infty} \frac{n!}{e^{n^2}} \text{ convergent by Ratio test.}$$

Q31/ $a_k = \dfrac{5^k}{3^k + 4^k}$, $b_k = \dfrac{5^k}{4^k}$

$$\frac{a_k}{b_k} = \frac{20^k}{20^k + 15^k} = \frac{1}{1 + \left(\frac{15}{20}\right)^k} \to 1$$

as $k \to \infty$ $\left(\left| \frac{15}{20} \right| < 1 \right)$

$$\frac{5}{4} > 1 \Rightarrow \sum_{k=1}^{\infty} \left(\frac{5}{4}\right)^k \text{ divergent}$$

$$\Rightarrow \sum_{k=1}^{\infty} \frac{5^k}{3^k + 4^k} \text{ divergent by Limit}$$

comparison test.