Homework 5 Solutions

§11.1 Sequences

Q6/ $\left\{a_n = \cos\frac{n\pi}{2}\right\} = \{0, -1, 0, 1, 0, \dots\}$

Q10/ $a_1 = 6$, $a_{n+1} = \frac{a_n}{n}$ $\Rightarrow$ $a_2 = 6$, $a_3 = 3$, $a_4 = 1$, $a_5 = \frac{1}{4}$.

Q23/ $a_n = \dfrac{3 + 5n^2}{n + n^2} = \dfrac{\frac{3}{n^2} + 5}{\frac{1}{n} + 1} \to 5$ as $n \to \infty$ (convergent)

Q26/ $a_n = 2 + (0.86)^n \to 2$ as $n \to \infty$ (as $|0.86| < 1$)

cos is continuous

Q32/ $\lim_{n\to\infty} \left\{\cos\left(\frac{n\pi}{n+1}\right)\right\} = \cos\left(\lim_{n\to\infty}\left\{\frac{n\pi}{n+1}\right\}\right)$

$= \cos\left(\lim_{n\to\infty}\left\{\frac{\pi}{1+\frac{1}{n}}\right\}\right) = \cos(\pi) = -1$

Q51/ $\lim_{x\to\infty} \arctan(x) = \frac{\pi}{2}$

$\lim_{n\to\infty} \{\ln(n)\} = \infty$

$\Rightarrow \lim_{n\to\infty} \{\arctan(\ln(n))\} = \frac{\pi}{2}$

Q64/ a) $a_1 = 1$, $a_2 = 3$, $a_3 = 1$, $a_4 = 3$, ...

$\Rightarrow \{a_n\} = \{1, 3, 1, 3, 1, 3, \dots\}$

$\Rightarrow$ divergent

b) $a_1 = 2 \Rightarrow a_2 = 2 \Rightarrow a_3 = 2 \Rightarrow$

$\{a_n\} = \{2, 2, 2, 2, \dots\}$

$\Rightarrow$ convergent

Q79/ $a_1 = \sqrt{2}$, $a_2 = \sqrt{2\sqrt{2}}$, $a_3 = \sqrt{2\sqrt{2\sqrt{2}}}$, ...

$\Rightarrow a_{n+1} = \sqrt{2a_n}$

Claim $a_{n+1} > a_n$

Assume $a_{k+1} > a_k \Rightarrow \sqrt{2a_{k+1}} > \sqrt{2a_k}$

$\Rightarrow a_{k+2} > a_{k+1}$

$a_2 > a_1 \Rightarrow a_3 > a_2 \Rightarrow a_4 > a_3 \Rightarrow \dots$

Claim $a_n < 2$

Assume $a_k < 2 \Rightarrow \sqrt{2a_k} < \sqrt{2 \cdot 2} = 2$

$\Rightarrow \quad a_{k+1} < 2$

$a_1 < 2 \Rightarrow a_2 < 2 \Rightarrow a_3 < 2 \quad \ldots$

$\Rightarrow$ $\{a_n\}$ monotone and bounded $\Rightarrow$ convergent

$a_n \to L$ as $n \to \infty$

$a_{n+1} = \sqrt{2a_n} \Rightarrow L = \sqrt{2L}$ ($a_n > 0$ all)

$\Rightarrow L^2 = 2L \Rightarrow L = 0$ or 2

$L \neq 0$ as $a_1 = \sqrt{2}$ and $\{a_n\}$ increasing

$\Rightarrow \quad L = 2.$

§11.2 Series

Q2/ $S_n = a_1 + a_2 + \dots + a_n$

$$\sum_{n=1}^{\infty} a_n = 5 \Rightarrow \lim_{n\to\infty} \{s_n\} = 5$$

↖ Limit of sequence

Q17/ $a = 3$

$r = \frac{-4}{3} \Rightarrow |r| = \frac{4}{3} > 1 \Rightarrow$ <u>divergent</u>.

Q23/ $$\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n} = \sum_{n=1}^{\infty} \frac{1}{4}\left(\frac{-3}{4}\right)^{n-1}$$

$a = \frac{1}{4}$, $r = \frac{-3}{4} \Rightarrow |r| = \frac{3}{4} < 1 \Rightarrow$ conv.

$$\sum_{n=1}^{\infty} \frac{(-3)^{n-1}}{4^n} = \frac{\frac{1}{4}}{1 - \frac{-3}{4}} = \frac{\frac{1}{4}}{\frac{7}{4}} = \frac{1}{7}.$$

Q37/ $$\lim_{n\to\infty} \left\{ \ln\left(\frac{n^2+1}{2n^2+1}\right) \right\} = \ln\left(\lim_{n\to\infty} \left\{ \frac{n^2+1}{2n^2+1} \right\} \right)$$

↑ ln continuous

$$= \ln\left(\lim_{n\to\infty} \left\{ \frac{1+\frac{1}{n^2}}{2+\frac{1}{n^2}} \right\} \right) = \ln\left(\frac{1}{2}\right) \neq 0$$

$\Rightarrow \sum_{n=1}^{\infty} \ln\left(\frac{n^2+1}{2n^2+1}\right)$ <u>divergent</u>.

Q43 $\dfrac{2}{n^2-1} = \dfrac{1}{n-1} - \dfrac{1}{n+1}$

$\Rightarrow S_1 = 1 - \frac{1}{3}$

$S_2 = 1 - \frac{1}{3} + \frac{1}{2} - \frac{1}{4}$

$S_3 = 1 - \cancel{\frac{1}{3}} + \frac{1}{2} - \frac{1}{4} + \cancel{\frac{1}{3}} - \frac{1}{5}$

$= 1 + \frac{1}{2} - \frac{1}{4} - \frac{1}{5}$

$S_4 = 1 + \frac{1}{2} - \cancel{\frac{1}{4}} - \frac{1}{5} + \cancel{\frac{1}{4}} - \frac{1}{6}$

$= 1 + \frac{1}{2} - \frac{1}{5} - \frac{1}{6}$

$\vdots$

$S_n = 1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2} \longrightarrow \frac{3}{2}$ as

$n \to \infty \Rightarrow$

$$\sum_{n=2}^{\infty} \frac{2}{n^2-1} = \frac{3}{2}$$ (convergent)

Q57/ $\displaystyle\sum_{n=1}^{\infty} (-5)^n x^n = \sum_{n=1}^{\infty} (-5x)(-5x)^{n-1}$ ← geo. series

convergent $\Leftrightarrow |-5x| < 1 \Leftrightarrow 5|x| < 1$

$\Leftrightarrow |x| < \frac{1}{5}$

$$|x| < \frac{1}{5} \Rightarrow \sum_{n=1}^{\infty} (-5x)(-5x)^{n-1} = \frac{-5x}{1+5x}$$

Q64/ $\ln\left(1+\frac{1}{n}\right) = \ln\left(\frac{n+1}{n}\right)$

$$= \ln(n+1) - \ln(n)$$

$$s_1 = \ln(2) - \ln(1) = \ln(2)$$

$$s_2 = \ln(3) - \cancel{\ln(2)} + \cancel{\ln(2)} - \ln(1)$$

$$= \ln(3)$$

$\vdots$

$$s_n = \ln(n+1) \longrightarrow \infty \quad \text{as} \quad n \to \infty$$

$$\Rightarrow \sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n}\right) \text{ divergent.}$$

Q81

$$0+0+0+\dots = \lim_{n\to\infty} \{s_n\}$$

where $s_n = \underbrace{0+0+\dots+0}_{n \text{ times}} = 0$

$\Rightarrow$ $0+0+0+\dots$ convergent with sum 0

$$1-1+1-1\dots = \lim_{n\to\infty} \{s_n\}$$

where $s_n = 1-1+1-1\dots + (-1)^{n-1} = \begin{cases} 1 & n \text{ odd} \\ 0 & n \text{ even} \end{cases}$

$\{1,0,1,0,1,0\dots\}$ divergent

$\Rightarrow$ $1-1+1-1\dots$ divergent $\Rightarrow$

$$0+0+0+0\dots \neq 1-1+1-1+\dots$$

§11.3 The Integral Test

Q5/ $f(x) = \frac{2}{5x-1} \Rightarrow f'(x) = \frac{-10}{(5x-1)^2} < 0$

$\Rightarrow f(x)$ decreasing on $[1, \infty)$

$5x-1 \neq 0$ on $[1, \infty) \Rightarrow$ continuous on $[1, \infty)$

$5x-1 > 0$ on $[1,\infty) \Rightarrow f(x) > 0$ on $[1, \infty)$

$$\int_1^\infty \frac{2}{5x-1}\,dx = \lim_{t\to\infty} \int_1^t \frac{2}{5x-1}\,dx$$

$$= \lim_{t\to\infty} \frac{2}{5}\ln|5x-1|\Big|_1^t$$

$$= \lim_{t\to\infty} \frac{2}{5}\ln|5t-1| - \frac{2}{5}\ln(4)$$

$$= \infty \quad \text{(divergent)}$$

$\Rightarrow \sum_{n=1}^{\infty} \frac{2}{5n-1}$ <u>divergent</u>.

Q13/ $a_n = \frac{1}{4n-1}$, $f(x) = \frac{1}{4x-1}$

$\Rightarrow f'(x) = \frac{-4}{(4x-1)^2} < 0$ on $[1, \infty)$

$\Rightarrow f(x)$ decreasing on $[1, \infty)$

$4x-1 \neq 0$ on $[1, \infty) \Rightarrow f(x)$ cts. on $[1, \infty)$

$4x-1 > 0$ on $[1, \infty) \Rightarrow f(x) > 0$ on $[1, \infty)$

$$\int_1^\infty \frac{1}{4x-1}\, dx = \lim_{t\to\infty} \frac{1}{4} \ln|4x-1| \Big|_1^t$$

$$= \lim_{t\to\infty} \left(\frac{1}{4} \ln|4t-1| - \frac{1}{4}\ln(3) \right)$$

$= \infty$ (divergent)

$\Rightarrow \frac{1}{3} + \frac{1}{7} + \frac{1}{11} + \ldots$ divergent

Q21/ $f(x) = \dfrac{1}{x\ln(x)}$

x, $\ln(x)$ increasing on $[2, \infty) \Rightarrow f(x)$ decreasing.

x, $\ln(x) \neq 0$ on $[2, \infty) \Rightarrow f(x)$ cts.

$x, \ln(x) > 0$ on $[2, \infty) \Rightarrow f(x) > 0$

$u = \ln(x) \Rightarrow \dfrac{du}{dx} = \dfrac{1}{x} \Rightarrow dx = x\,du$

$$\Rightarrow \int \frac{1}{x\ln(x)}\, dx = \int \frac{1}{u}\, du = \ln|u| + c$$

$$= \ln|\ln(x)| + C$$

$$\Rightarrow \int_2^\infty \frac{1}{x\ln(x)}\,dx = \lim_{t\to\infty}\left(\ln|\ln(t)| - \ln|\ln(2)|\right)$$

$$= \infty$$

$$\Rightarrow \sum_{n=2}^\infty \frac{1}{n\ln(n)} \quad \underline{\text{divergent}}.$$

Q27/ $\cos \pi n = \begin{cases} 1 & n \text{ even} \\ -1 & n \text{ odd} \end{cases}$

$$\Rightarrow \frac{\cos \pi n}{\sqrt{n}} = \begin{cases} \frac{1}{\sqrt{n}} & n \text{ even} \\ \frac{-1}{\sqrt{n}} & n \text{ odd} \end{cases}$$

$\Rightarrow$ Cannot use integral test as terms are <u>not</u> eventually $+$.

Q29/ $f(x) = \dfrac{1}{x\ln(x)^p}$ $(x \geq 2)$ $\Rightarrow f'(x) = \dfrac{-\left(\ln(x)^p + p\ln(x)^{p-1}\right)}{\left(x\ln(x)^p\right)^2}$

$$= \frac{-\left(\ln(x) + p\right)\ln(x)^{p-1}}{\left(x\ln(x)^p\right)^2}$$

Because $\ln(x) \to \infty$ as $x \to \infty$, this function is eventually negative for any value of p.

$\Rightarrow$ $f(x)$ eventually decreasing

$x,\ \ln(x)^p \neq 0$ for all x in $[2,\infty)$ $\Rightarrow$

$f(x)$ cts.

$x,\ \ln(x)^p > 0$ on $[2,\infty)$ $\Rightarrow$ $f(x) > 0$ on $[2,\infty)$

$$\int \frac{1}{x\ln(x)^p}\,dx = \begin{cases} \frac{1}{1-p}\ln(x)^{1-p} + C & p \neq 1 \\ \ln|\ln(x)| + C & p = 1 \end{cases}$$

$$\Rightarrow \int_2^\infty \frac{1}{x\ln(x)^p}\,dx = \begin{cases} \lim\limits_{t\to\infty}\left(\frac{1}{1-p}\ln(t)^{1-p} - \frac{1}{1-p}\ln(2)^{1-p}\right) & p\neq 1 \\ \lim\limits_{t\to\infty}\left(\ln|\ln(t)| - \ln|\ln(2)|\right) & p = 1 \end{cases}$$

$$= \begin{cases} \frac{-1}{1-p}\ln(2)^{1-p} & \text{if } 1-p<0 \Leftrightarrow p>1 \\ \text{div} & \text{if } 1-p>0 \Leftrightarrow p<1 \\ \text{div} & \text{if } p=1 \end{cases}$$

$$\Rightarrow \sum_{n=2}^{\infty} \frac{1}{n\ln(n)^p} = \begin{cases} \text{conv.} & p>1 \\ \text{div} & p\leq 1 \end{cases}$$

Q33 $\quad \zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$ $\leftarrow$ convergent $\Leftrightarrow$ $x > 1$

$\Rightarrow$ Domain of ζ is $(1, \infty)$

Q34

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \dots = \frac{\pi^2}{6}$$

HARD!

a) $\displaystyle \sum_{n=2}^{\infty} \frac{1}{n^2} = \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right) - 1 = \frac{\pi^2}{6} - 1$

b) $\displaystyle \sum_{n=3}^{\infty} \frac{1}{(n+1)^2} = \frac{1}{4^2} + \frac{1}{5^2} + \dots$

$$= \left(\sum_{n=1}^{\infty} \frac{1}{n^2}\right) - 1 - \frac{1}{2^2} - \frac{1}{3^2}$$

$$= \frac{\pi^2}{6} - 1 - \frac{1}{4} - \frac{1}{9}$$

c) $\displaystyle \sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \sum_{n=1}^{\infty} \frac{1}{4n^2} = \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{24}.$

§ 11.4 Comparison Tests

Q 6/ $a_n = \dfrac{n-1}{n^3+1}$, $b_n = \dfrac{1}{n^2}$

$$\frac{a_n}{b_n} = \frac{n^3 - n^2}{n^3+1} = \frac{1 - \frac{1}{n}}{1 + \frac{1}{n^3}} \longrightarrow 1 > 0$$

as $n \longrightarrow \infty$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \text{ conv } \underset{\substack{\text{Limit} \\ \text{C.T.}}}{\Longrightarrow} \sum_{n=1}^{\infty} \frac{n-1}{n^3+1} \text{ conv.}$$

Q7/ $a_n = \dfrac{9^n}{3 + 10^n}$ $b_n = \dfrac{9^n}{10^n} = \left(\dfrac{9}{10}\right)^n$

$$\frac{a_n}{b_n} = \frac{90^n}{3 \cdot 9^n + 90^n} = \frac{1}{3 \cdot \left(\frac{9}{90}\right)^n + 1} \longrightarrow 1 > 0$$

as $n \longrightarrow \infty$

$$\sum_{n=1}^{\infty} \left(\frac{9}{10}\right)^n \text{ conv } \underset{\substack{\text{Limit} \\ \text{C.T}}}{\Longrightarrow} \sum_{n=1}^{\infty} \frac{9^n}{3 + 10^n} \text{ conv.}$$

Q12/ $a_k = \dfrac{(2k-1)(k^2-1)}{k+1\ (k^2+4)^2} = \dfrac{(2k-1)(k-1)}{(k^2+4)^2}$

$b_k = \dfrac{1}{k^2} \Rightarrow \dfrac{a_k}{b_k} = \dfrac{(2k^2-k)(k^2-k)}{(k^2+4)^2}$

$= \dfrac{\left(2-\frac{1}{k}\right)\left(1-\frac{1}{k}\right)}{\left(1+\frac{4}{k^2}\right)^2} \rightarrow 2 > 0$

as $k \to \infty$

$\sum_{k=1}^{\infty} \dfrac{1}{k^2}$ conv. $\underset{\text{Limit C.T.}}{\Rightarrow}$ $\sum_{k=1}^{\infty} \dfrac{(2k-1)(k^2-1)}{k+1\ (k^2+4)^2}$ conv.

Q13/ $0 \leq 1 + \cos(n) \leq 2$

$\Rightarrow 0 \leq \dfrac{1+\cos(n)}{e^n} \leq \dfrac{2}{e^n}$

$\sum_{n=1}^{\infty} \dfrac{2}{e^n} = \sum_{n=1}^{\infty} 2 \cdot \left(\frac{1}{e}\right)^n$ conv. as $\left|\frac{1}{e}\right| < 1$

$\Rightarrow \sum_{n=1}^{\infty} \dfrac{1+\cos(n)}{e^n}$ conv.

Q16/ $a_n = \frac{1}{n^n}$, $b_n = \frac{1}{n^2}$

If $n \geq 2 \Rightarrow n^n \geq n^2 > 0$

$\Rightarrow 0 < \frac{1}{n^n} \leq \frac{1}{n^2}$ for $n \geq 2$

$\sum_{n=2}^{\infty} \frac{1}{n^2}$ conv $\Rightarrow \sum_{n=2}^{\infty} \frac{1}{n^n}$ conv

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^n}$ conv.

Q31/ $a_n = \sin\left(\frac{1}{n}\right)$, $b_n = \frac{1}{n}$

$$\frac{a_n}{b_n} = \frac{\sin\left(\frac{1}{n}\right)}{\frac{1}{n}}$$

Let $f(x) = \frac{\sin(1/x)}{1/x}$

$$\lim_{x \to \infty} f(x) \underset{\text{L'Hospitale}}{=} \lim_{x \to \infty} \frac{\frac{-1}{x^2}\cos(1/x)}{\frac{-1}{x^2}}$$

$$= \lim_{x \to \infty} \cos(1/x)$$

$$= \cos\left(\lim_{x \to \infty} 1/x\right) = \cos(0) = 1 > 0$$

$$\Rightarrow \lim_{n\to\infty} \frac{a_n}{b_n} = 1 > 0$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ div} \underset{\substack{\text{Limit}\\ \text{C.T.}}}{\Rightarrow} \sum_{n=1}^{\infty} \sin(1/n) \text{ div.}$$

Q32/ $a_n = \frac{1}{n^{(1+1/n)}}$ $b_n = \frac{1}{n}$

$$\frac{a_n}{b_n} = \left(\frac{n^{1+1/n}}{n}\right)^{-1} = n^{-1/n}$$

Let $f(x) = x^{-1/x} \Rightarrow \ln(f(x)) = -\frac{\ln(x)}{x}$

$$\lim_{x\to\infty} \ln(f(x)) \underset{\text{L'Hopitale}}{=} \lim_{x\to\infty} \left(\frac{-\frac{1}{x}}{1}\right) = 0$$

$$\Rightarrow \lim_{x\to\infty} f(x) = e^0 = 1$$

$$\Rightarrow \lim_{n\to\infty} \left\{\frac{a_n}{b_n}\right\} = 1 > 0$$

$$\sum_{n=1}^{\infty} \frac{1}{n} \text{ div} \underset{\substack{\text{Limit}\\ \text{C.T.}}}{\Rightarrow} \sum_{n=1}^{\infty} \frac{1}{n^{1+1/n}} \text{ div.}$$

Q37/ $\quad 0.d_1d_2d_3\ldots = \sum_{n=1}^{\infty} \frac{d_n}{10^n} \qquad (0 \le d_n \le 9)$

$\Rightarrow \quad 0 \le \frac{d_n}{10^n} \le \frac{9}{10^n}$ for all $n \geq 1$

$$\sum_{n=1}^{\infty} \frac{9}{10^n} = \frac{\frac{9}{10}}{1-\frac{1}{10}} = 1 \leftarrow$$ convergent geometric series $\left(r = \frac{1}{10} \right)$

$\underset{\text{C.T.}}{\Rightarrow} \quad \sum_{n=1}^{\infty} \frac{d_n}{10^n}$ convergent

Q40/

a) $\quad a_n, b_n \geq 0$ for all $n \geq 1$

$\lim_{n\to\infty} \left\{ \frac{a_n}{b_n} \right\} = 0 \quad \Rightarrow$ There exists

$N > 0$ such that $\quad 0 \le \frac{a_n}{b_n} \le 1$

for all $n > N$.

$\Rightarrow \quad 0 \le a_n \le b_n$ for all $n > N$

$\sum_{n=1}^{\infty} b_n$ conv. $\Rightarrow \sum_{n=1}^{\infty} a_n$ conv.

b) i) $a_n = \dfrac{\ln(n)}{n^3}$, $b_n = \dfrac{1}{n^2}$

$$\frac{a_n}{b_n} = \frac{\ln(n)}{n} \quad , \quad f(x) = \frac{\ln(x)}{x}$$

$$\lim_{x \to \infty} \frac{\ln(x)}{x} = \lim_{x \to \infty} \frac{\frac{1}{x}}{1} = 0$$

L'Hospitale

$$\Rightarrow \quad \frac{a_n}{b_n} \longrightarrow 0 \quad \text{as } n \to \infty$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \text{ conv.} \quad \Rightarrow \quad \sum_{n=1}^{\infty} \frac{\ln(n)}{n^3} \text{ conv.}$$

ii) $a_n = \dfrac{\ln(n)}{\sqrt{n}\, e^n}$, $b_n = \dfrac{1}{e^n}$

$$\frac{a_n}{b_n} = \frac{\ln(n)}{\sqrt{n}} \quad , \quad f(x) = \frac{\ln(x)}{\sqrt{x}}$$

$$\lim_{x \to \infty} \frac{\ln(x)}{\sqrt{x}} = \lim_{x \to \infty} \frac{\frac{1}{x}}{\frac{1}{2}\frac{1}{\sqrt{x}}} = \lim_{x \to \infty} \frac{1}{2\sqrt{x}}$$

$$= 0$$

$$\Rightarrow \lim_{n \to \infty} \left\{\frac{a_n}{b_n}\right\} = 0 \quad , \quad \left|\frac{1}{e}\right| < 1 \quad \Rightarrow$$

$$\sum_{n=1}^{\infty} \frac{1}{e^n} \text{ conv.} \quad \Rightarrow \quad \sum_{n=1}^{\infty} \frac{\ln(n)}{\sqrt{n}\, e^n} \text{ conv.}$$