1B Homework 2 Solutions

§7.4

Q1/ $$\frac{A}{1+2x} + \frac{B}{3-x}$$

Q6/ a) $$t^6+t^3 = t^3(t^3+1) = t^3(t+1)(t^2-t+1)$$

$$\Rightarrow \frac{t^6+1}{t^6+t^3} = 1 + \frac{A}{t} + \frac{B}{t^2} + \frac{C}{t^3} + \frac{D}{t+1} + \frac{Et+F}{t^2-t+1}$$

b) $$x^4+2x^2+1 = (x^2+1)^2$$

$$x^2-x = x(x-1)$$

$$\Rightarrow \frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+1} + \frac{Ex+F}{(x^2+1)^2}$$

Q10/ $$\frac{y}{(y+4)(2y-1)} = \frac{A}{y+4} + \frac{B}{2y-1} = \frac{A(2y-1)+B(y+4)}{(y+4)(2y-1)}$$

$$= \frac{(2A+B)y+(4B-A)}{(y+4)(2y-1)} \Rightarrow \begin{matrix} 2A+B=1 \\ 4B-A=0 \end{matrix} \Rightarrow A = 4B$$

$$\Rightarrow 9B = 1 \Rightarrow B = \frac{1}{9}, \quad A = \frac{4}{9}$$

$$\Rightarrow \int \frac{y}{(y+4)(2y-1)}\,dy = \frac{4}{9}\int \frac{1}{y+4}\,dy + \frac{1}{9}\int \frac{1}{2y-1}\,dy$$

$$= \frac{4}{9}\ln|y+4| + \frac{1}{18}\ln|2y-1| + C$$

Q23/ $\frac{10}{(x-1)(x^2+9)} = \frac{A}{(x-1)} + \frac{Bx+C}{x^2+9} = \frac{A(x^2+9)+(Bx+C)(x-1)}{(x-1)(x^2+9)}$

$= \frac{(A+B)x^2 + (C-B)x + (9A-C)}{(x-1)(x^2+9)}$

$\Rightarrow \begin{matrix} A+B=0 \\ C-B=0 \\ 9A-C=10 \end{matrix} \Rightarrow \begin{matrix} A=-B \\ B=C \end{matrix} \Rightarrow A=-C \Rightarrow 10A=10 \Rightarrow \begin{matrix} A=1 \\ B=-1 \\ C=-1 \end{matrix}$

$\Rightarrow \int \frac{10}{(x-1)(x^2+9)} dx = \int \frac{1}{x-1} dx - \int \frac{x+1}{x^2+9} dx$

$= \ln|x-1| - \int \frac{x}{x^2+9} dx - \int \frac{1}{x^2+9} dx$

$= \ln|x-1| - \frac{1}{2} \ln|x^2+9| - \frac{1}{3} \arctan\left(\frac{x}{3}\right) + C$

Q37/ (Hard)

$\frac{x^2-3x+7}{(x^2-4x+6)^2} = \frac{Ax+B}{(x^2-4x+6)} + \frac{Cx+D}{(x^2-4x+6)^2}$

$= \frac{(Ax+B)(x^2-4x+6) + Cx+D}{(x^2-4x+6)^2}$

$= \frac{Ax^3 + (B-4A)x^2 + (6A-4B+C)x + (D+6B)}{(x^2-4x+6)^2}$

$\Rightarrow \begin{matrix} A=0 \\ B-4A=1 \\ 6A-4B+C=-3 \\ D+6B=7 \end{matrix} \Rightarrow \begin{matrix} A=0 \\ B=1 \\ C=1 \\ D=1 \end{matrix}$

$$\Rightarrow \int \frac{x^2 - 3x + 7}{(x^2 - 4x + 6)^2}\, dx$$

$$= \int \frac{1}{(x-2)^2 + 2}\, dx + \int \frac{(x-2)}{((x-2)^2+2)^2}\, dx + 3\int \frac{1}{((x-2)^2+2)^2}\, dx$$

(Let $u = x - 2 \Rightarrow \frac{du}{dx} = 1 \Rightarrow dx = du$)

$$= \int \frac{1}{u^2+2}\, du + \int \frac{u}{(u^2+2)^2}\, du + 3\int \frac{1}{(u^2+2)^2}\, du$$

$$= \frac{1}{\sqrt{2}} \arctan\left(\frac{u}{\sqrt{2}}\right) - \frac{1}{2(u^2+2)} + 3\int \frac{1}{(u^2+2)^2}\, du$$

Harder.

Look at 72/. Or last example in I.B.P. lecture

$$\int \frac{1}{(u^2+2)^2}\, du = \frac{u}{4(u^2+2)} + \frac{1}{4}\int \frac{1}{u^2+2}\, du$$

$$= \frac{u}{4(u^2+2)} + \frac{1}{4} \cdot \frac{1}{\sqrt{2}} \arctan\left(\frac{u}{\sqrt{2}}\right) + C$$

$$\Rightarrow \int \frac{x^2 - 3x + 7}{(x^2-4x+6)^2}\, dx = \frac{1}{\sqrt{2}} \arctan\left(\frac{x-2}{\sqrt{2}}\right) - \frac{1}{2(x^2-4x+6)} + \frac{3(x-2)}{4(x^2-4x+6)} + \frac{3}{4\cdot\sqrt{2}} \arctan\left(\frac{x-2}{\sqrt{2}}\right) + C$$

Q51/ Let $u = 1 + e^x \Rightarrow \frac{du}{dx} = e^x = u - 1 \Rightarrow dx = \frac{du}{u-1}$

$$\Rightarrow \int \frac{1}{1+e^x}\,dx = \int \frac{1}{u(u-1)}\,du$$

$$\frac{1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1} = \frac{A(u-1) + Bu}{u(u-1)} = \frac{(A+B)u - A}{u(u-1)}$$

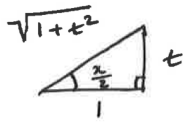
$$\Rightarrow \begin{matrix} -A = 1 \\ A + B = 0 \end{matrix} \Rightarrow A = -1,\ B = 1$$

$$\Rightarrow \int \frac{1}{1+e^x}\,dx = \int \frac{-1}{u}\,du + \int \frac{1}{u-1}\,du = -\ln|u| + \ln|u-1| + C$$

$$= \ln\left|\frac{u-1}{u}\right| + C = \ln\left|\frac{e^x}{1+e^x}\right| + C$$

Q59/

a) $t = \tan\left(\frac{x}{2}\right) \qquad -\pi < x < \pi$



$$\Rightarrow \cos\left(\frac{x}{2}\right) = \frac{1}{\sqrt{1+t^2}}$$

$$\sin\left(\frac{x}{2}\right) = \frac{t}{\sqrt{1+t^2}}$$

b) $\cos(x) = \cos^2\left(\frac{x}{2}\right) - \sin^2\left(\frac{x}{2}\right) = \frac{1}{1+t^2} - \frac{t^2}{1+t^2}$

$$= \frac{1-t^2}{1+t^2}$$

$$\sin(x) = 2\cos\left(\frac{x}{2}\right)\sin\left(\frac{x}{2}\right) = \frac{2t}{1+t^2}$$

c) $\frac{dt}{dx} = \frac{1}{2}\sec^2\left(\frac{x}{2}\right) = \frac{1}{2\cos^2\left(\frac{x}{2}\right)} = \frac{1+t^2}{2}$

$$\Rightarrow dx = \frac{2}{1+t^2}\,dt$$

Q72 $f(x) = \frac{1}{(x^2+a^2)^{n-1}}$, $g'(x) = 1$

$f'(x) = \frac{-2(n-1)x}{(x^2+a^2)^n}$, $g(x) = x$

$$\Rightarrow \int \frac{1}{(x^2+a^2)^{n-1}}\,dx = \frac{x}{(x^2+a^2)^{n-1}} + 2(n-1)\int \frac{x^2}{(x^2+a^2)^n}\,dx$$

$$= \frac{x}{(x^2+a^2)^{n-1}} + 2(n-1)\int \frac{(x^2+a^2) - a^2}{(x^2+a^2)^n}\,dx$$

$$= \frac{x}{(x^2+a^2)^{n-1}} + 2(n-1)\int \frac{1}{(x^2+a^2)^{n-1}}\,dx - 2a^2(n-1)\int \frac{1}{(x^2+a^2)^n}\,dz$$

$$\Rightarrow \int \frac{1}{(x^2+a^2)^n}\,dx = \frac{\left(\int \frac{1}{(x^2+a^2)^{n-1}}\,dx - \frac{x}{(x^2+a^2)^{n-1}} - 2(n-1)\int \frac{1}{(x^2+a^2)^{n-1}}\,dx\right)}{-2a^2(n-1)}$$

$$= \frac{x}{2a^2(n-1)(x^2+a^2)^{n-1}} + \frac{2n-3}{2a^2(n-1)}\int \frac{1}{(x^2+a^2)^{n-1}}\,dx$$

b) $$\int \frac{1}{(x^2+1)^2}\,dx = \frac{x}{2(x^2+1)} + \frac{1}{2}\int \frac{1}{x^2+1}\,dx$$

$$= \frac{x}{2(x^2+1)} + \frac{1}{2}\arctan(x) + C$$

$$\int \frac{1}{(x^2+1)^3}\,dx = \frac{x}{4(x^2+1)^2} + \frac{3}{4}\int \frac{1}{(x^2+1)^2}\,dx$$

$$= \frac{x}{4(x^2+1)^2} + \frac{3x}{8(x^2+1)} + \frac{3}{8}\arctan(x) + C$$

§7.2

Q2 $u = \cos\theta \Rightarrow \dfrac{du}{d\theta} = -\sin\theta \Rightarrow d\theta = \dfrac{du}{-\sin\theta}$

$$\Rightarrow \int \sin^3\theta \cos^4\theta \, d\theta = -\int (1-u^2)u^4 \, du = -\frac{1}{5}u^5 + \frac{1}{7}u^7 + C$$

$$= -\frac{1}{5}\cos^5\theta + \frac{1}{7}\cos^7\theta + C$$

Q6 $u = t^2 \Rightarrow \dfrac{du}{dt} = 2t \Rightarrow dt = \dfrac{du}{2t}$

$$\Rightarrow \int t\cos^5(t^2)\, dt = \frac{1}{2}\int \cos^5(u)\, du$$

$x = \sin(u) \Rightarrow \dfrac{dx}{du} = \cos(u) \Rightarrow du = \dfrac{dx}{\cos(u)}$

$$\Rightarrow \frac{1}{2}\int \cos^5(u)\, du = \frac{1}{2}\int (1-x^2)^2 \, dx = \frac{1}{2}x + \frac{-1}{3}x^3 + \frac{1}{10}x^5 + C$$

$$= \frac{1}{2}\sin(u) - \frac{1}{3}\sin^3(u) + \frac{1}{10}\sin^5(u) + C$$

$$= \frac{1}{2}\sin(t^2) - \frac{1}{3}\sin^3(t^2) + \frac{1}{10}\sin^5(t^2) + C$$

Q7 $$\int_0^{\frac{\pi}{2}} \cos^2\theta \, d\theta = \int_0^{\pi/2} \frac{1+\cos(2\theta)}{2}\, d\theta = \frac{\theta}{2} + \frac{\sin(2\theta)}{4}\Big|_0^{\frac{\pi}{2}}$$

$$= \frac{\pi}{4}$$

Q15 $$\int \cot x \cos^2 x \, dx = \int \sin^{-1}x \cos^3 x \, dx$$

$u = \sin x \Rightarrow \dfrac{du}{dx} = \cos x \Rightarrow dx = \dfrac{du}{\cos x}$

$$\Rightarrow \int \cot x \cos^2 x \, dx = \int u^{-1}(1-u^2)\, du = \ln|u| - \frac{1}{2}u^2 + C$$

$$= \ln|\sin x| - \frac{1}{2}\sin^2 x + C$$

Q27/ $\int \tan^3(x)\sec(x)\,dx = \int \sin^3(x)\cos^{-4}(x)\,dx$

$u = \cos(x) \Rightarrow \frac{du}{dx} = -\sin(x) \Rightarrow dx = \frac{du}{-\sin(x)}$

$\Rightarrow -\int \sin^2(x)\cos^{-4}(x)\,du = -\int (1-u^2)u^{-4}\,du = -\int u^{-4} - u^{-2}\,du$

$= \frac{1}{3}u^{-3} - u^{-1} + C = \frac{1}{3}\cos^{-3}(x) - \cos^{-1}(x) + C$

Q29/ $\int \tan^3(x)\sec^6(x)\,dx$

$u = \tan(x) \Rightarrow \frac{du}{dx} = \sec^2(x) \Rightarrow dx = \frac{du}{\sec^2(x)}$

$\Rightarrow \int \tan^3(x)\sec^6(x)\,dx = \int \tan^3(x)\sec^4(x)\,du = \int u^3(1+u^2)^2\,du$

$= \int u^3 + 2u^5 + u^7\,du = \frac{1}{4}u^4 + \frac{2}{6}u^6 + \frac{1}{8}u^8 + C$

$= \frac{1}{4}\tan^4(x) + \frac{1}{3}\tan^6(x) + \frac{1}{8}\tan^8(x) + C$

48/ $\int \frac{1}{\cos x - 1}\,dx = \int \frac{1+\cos x}{\cos^2 x - 1}\,dx = -\int \frac{1+\cos x}{\sin^2 x}\,dx$

$= -\int \csc^2 x\,dx - \int \sin^{-2}x\cos x\,dx$

$u = \sin x \Rightarrow \frac{du}{dx} = \cos x \Rightarrow dx = \frac{du}{\cos x}$

$\Rightarrow \int \sin^{-2}x\cos x\,dx = \int u^{-2}\,du = \frac{-1}{u} + C = \frac{-1}{\sin x} + C$

$\int \csc^2 x\,dx = -\cot x$

$\Rightarrow \int \frac{1}{\cos x - 1}\,dx = \cot x + \csc x + C$

Q58/ $0 \le x \le \frac{\pi}{4} \Rightarrow 0 \le \tan x \le 1 \Rightarrow \tan^2 x \le \tan x$

$$\Rightarrow \text{Area} = \int_0^{\frac{\pi}{4}} \tan x - \tan^2 x \, dx$$

$$\int_0^{\frac{\pi}{4}} \tan x \, dx = \ln|\sec(x)| \Big|_0^{\pi/4} = \ln|\sec(\pi/4)| - \ln|\sec(0)|$$

$$= \ln\sqrt{2}$$

let $u = \tan x \Rightarrow \frac{du}{dx} = \sec^2 x \Rightarrow dx = \frac{du}{\sec^2 x}$

$$\int_0^{\pi/4} \tan^2 x \, dx = \int_0^1 u^2 \cdot \frac{1}{u^2+1} \, du$$

$$= \int_0^1 \frac{u^2+1-1}{u^2+1} \, du = \int_0^1 1 - \frac{1}{u^2+1} \, du$$

$$= u - \arctan(u) \Big|_0^1 = \left(1 - \frac{\pi}{4}\right) - (0 - 0)$$

$$\Rightarrow \text{Area} = \ln\sqrt{2} - 1 + \frac{\pi}{4}$$

§7.3/

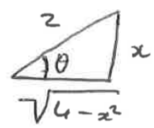
Q1/ $x = 2\sin\theta \Rightarrow \frac{dx}{d\theta} = 2\cos\theta \Rightarrow dx = 2\cos\theta \, d\theta$

$$\Rightarrow \int \frac{1}{x^2\sqrt{4-x^2}} \, dx = \int \frac{2\cos\theta}{2^2\sin^2\theta \cdot 2\cos\theta} \, d\theta = \frac{1}{4}\int \csc^2\theta \, d\theta$$

$$= \frac{-1}{4}\cot(\theta) + C$$

$$= -\frac{\sqrt{4-x^2}}{4x} + C$$

$\left(x = 2\sin\theta \Rightarrow \sin\theta = \frac{x}{2} \right.$



$\Rightarrow \cot(\theta) = \frac{\sqrt{4-x^2}}{x}$

Q2/ $x = 2\tan\theta \Rightarrow \frac{dx}{d\theta} = 2\sec^2\theta \Rightarrow dx = 2\sec^2\theta \, d\theta$

$$\Rightarrow \int \frac{x^3}{\sqrt{4+x^2}} \, dx = \int \frac{8\tan^3\theta \cdot 2\sec^2\theta}{2\sec\theta} \, d\theta$$

$$= 8\int \sin^3\theta \cos^{-1}\theta \, d\theta$$

$u = \cos(\theta) \Rightarrow \frac{du}{d\theta} = -\sin\theta \Rightarrow d\theta = \frac{-du}{\sin\theta}$

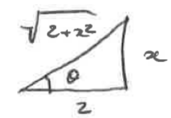
$$\Rightarrow 8\int \sin^3\theta \cos^{-1}\theta \, d\theta = -8\int (1-u^2)\, u^{-1} \, du$$

$$= -8\ln|u| + 4u^2 + C$$

$$= -8\ln|\cos\theta| + 4\cos^2\theta + C$$

$$= -8\ln\left|\frac{2}{\sqrt{2+x^2}}\right| + \frac{16}{2+x^2} + C$$

$\left(\tan\theta = \frac{x}{2} \right.$



$\left. \Rightarrow \cos\theta = \frac{2}{\sqrt{2+x^2}} \right)$

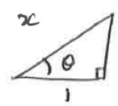
Q5/ $x = \sec\theta \Rightarrow \dfrac{dx}{d\theta} = \tan\theta\sec\theta \Rightarrow dx = \tan\theta\sec\theta\, d\theta$

$$\Rightarrow \int \frac{\sqrt{x^2-1}}{x^4}\,dx = \int \frac{\tan\theta}{\sec^4\theta}\tan\theta\sec\theta\, d\theta = \int \sin^2\theta\cos\theta\, d\theta$$

$u = \sin\theta \Rightarrow \dfrac{du}{d\theta} = \cos\theta \Rightarrow d\theta = \dfrac{du}{\cos\theta}$

$$\Rightarrow \int \sin^2\theta\cos\theta\, d\theta = \int u^2\, du = \frac{1}{3}u^3 + C = \frac{1}{3}\sin^3\theta + C$$

$\left(x = \sec\theta \Rightarrow \cos\theta = \dfrac{1}{x} \right.$



$\sqrt{x^2-1} \Rightarrow \sin\theta = \left.\dfrac{\sqrt{x^2-1}}{x}\right)$

$$\Rightarrow \int \frac{\sqrt{x^2-1}}{x^4}\,dx = \frac{(x^2-1)^{3/2}}{3x^3} + C$$

Q15/ $x = a\sin\theta \Rightarrow \dfrac{dx}{d\theta} = a\cos\theta \Rightarrow dx = a\cos\theta\, d\theta$

$\left(\frac{-\pi}{2} \leq \theta \leq \frac{\pi}{2}\right)$

$$\Rightarrow \int_0^a x^2\sqrt{a^2-x^2}\,dx = a^4\int_0^{\frac{\pi}{2}} \sin^2\theta\cos^2\theta\, d\theta$$

$$= \frac{a^4}{4}\int_0^{\frac{\pi}{2}} (\sin 2\theta)^2\, d\theta = \frac{a^4}{8}\int_0^{\frac{\pi}{2}} (1-\cos 4\theta)\, d\theta$$

$$= \frac{a^4}{8}\left(\theta - \frac{\sin 4\theta}{4}\right)\Bigg|_0^{\frac{\pi}{2}} = \frac{a^4\pi}{16}$$

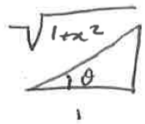
Q20/ $x = \tan\theta \Rightarrow \dfrac{dx}{d\theta} = \sec^2\theta \Rightarrow dx = \sec^2\theta\, d\theta$

$$\Rightarrow \int \frac{x}{\sqrt{1+x^2}}\,dx = \int \frac{\tan\theta\sec^2\theta}{\sec\theta}\,d\theta = \int \sin\theta\cos^{-2}\theta\, d\theta$$

$$u = \cos\theta \Rightarrow \frac{du}{d\theta} = -\sin\theta \Rightarrow d\theta = \frac{-du}{\sin\theta} \Rightarrow$$

$$\int \sin\theta \cos^{-2}\theta \, d\theta = -\int u^{-2} du = \frac{1}{u} + C = \frac{1}{\cos\theta} + C$$

$$\left(x = \tan\theta \Rightarrow \right.$$



$$\left. \Rightarrow \cos\theta = \frac{1}{\sqrt{1+x^2}} \right)$$

$$\Rightarrow \int \frac{x}{\sqrt{1+x^2}} dx = \sqrt{1+x^2} + C$$

Q28

$$\frac{x^2+1}{(x^2-2x+2)^2} = \frac{x^2+1}{((x-1)^2+1)^2}$$

Let $u = x-1 \Rightarrow du = dx \Rightarrow$

$$\int \frac{x^2+1}{(x^2-2x+2)^2} dx = \int \frac{(u+1)^2+1}{(u^2+1)^2} du = \int \frac{u^2+2u+2}{(u^2+1)^2} du$$

$$u = \tan\theta \Rightarrow \frac{du}{d\theta} = \sec^2\theta \Rightarrow du = \sec^2\theta \, d\theta$$

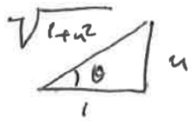
$$\Rightarrow \int \frac{u^2+2u+2}{(u^2+1)^2} du = \int \frac{\tan^2\theta + 2\tan\theta + 2}{\sec^4\theta} \cdot \sec^2\theta \, d\theta$$

$$= \int \sin^2\theta \, d\theta + 2\int \sin\theta\cos\theta \, d\theta + 2\int \cos^2\theta \, d\theta$$

$$= \int \frac{1-\cos 2\theta}{2} d\theta + \int \sin 2\theta \, d\theta + \int 1 + \cos(2\theta) \, d\theta$$

$$= \frac{3}{2}\theta - \frac{\sin 2\theta}{4} - \frac{\cos 2\theta}{2} + \frac{\sin 2\theta}{2} + C$$

$$= \frac{3}{2}\theta + \frac{1}{8}\sin\theta\cos\theta - \frac{1}{2}\left(\cos^2\theta - \sin^2\theta\right) + C$$

$\Big($ $u = \tan\theta \Rightarrow$ 

$\Rightarrow \sin\theta = \frac{u}{\sqrt{1+u^2}}$, $\cos\theta = \frac{1}{\sqrt{1+u^2}}$ $\Big)$

$$= \frac{3}{2}\arctan(u) + \frac{u}{8(1+u^2)} - \frac{1}{2}\left(\frac{1}{1+u^2} - \frac{u^2}{1+u^2}\right) + C$$

$$= \frac{3}{2}\arctan(x-1) + \frac{x-1}{8(x^2-2x+2)} - \frac{2x-x^2}{2(x^2-2x+2)} + C$$

Q30/ $u = \sin t \Rightarrow \frac{du}{dt} = \cos t \Rightarrow dt = \frac{du}{\cos t}$

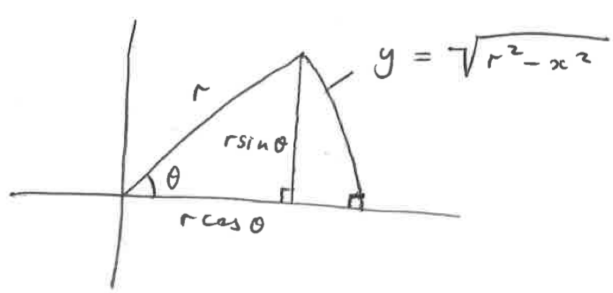
$$\Rightarrow \int_0^{\pi/2} \frac{\cos t}{\sqrt{1+\sin^2 t}}\,dt = \int_0^1 \frac{1}{\sqrt{1+u^2}}\,du$$

$u = \tan\theta \Rightarrow \frac{du}{d\theta} = \sec^2\theta \Rightarrow du = \sec^2\theta\,d\theta$

$$\Rightarrow \int_0^1 \frac{1}{\sqrt{1+u^2}}\,du = \int_0^{\pi/4} \sec\theta\,d\theta = \ln\left|\tan\theta + \sec\theta\right| \Big|_0^{\frac{\pi}{4}}$$

$$= \ln\left|1+\sqrt{2}\right|.$$

Q35/ (Hard)



$$\text{Area} = \frac{1}{2} r\cos\theta \, r\sin\theta + \int_{r\cos\theta}^{r} \sqrt{r^2 - x^2}\, dx$$

$$0 < \theta < \frac{\pi}{2}$$

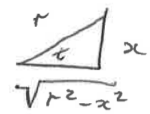
$$x = r\sin t \Rightarrow \frac{dx}{dt} = r\cos t \Rightarrow dx = r\cos t\, dt$$

$$\Rightarrow \int \sqrt{r^2 - x^2}\, dx = r^2 \int \cos^2 t\, dt = r^2 \int \frac{1 + \cos 2t}{2}\, dt$$

$$= \frac{r^2}{2}\left(t + \frac{\sin(2t)}{2} \right) + c$$

$$= \frac{r^2}{2}\left(t + \sin(t)\cos(t) \right)$$

$$\left(\frac{x}{r} = \sin(t) \Rightarrow \right.$$



$$\left. \right)$$

$$= \frac{r^2}{2}\left(\arcsin\left(\frac{x}{r}\right) + \frac{x}{r} \cdot \frac{\sqrt{r^2 - x^2}}{r} \right)$$

$$\Rightarrow \int_{r\cos\theta}^{r} \sqrt{r^2 - x^2}\, dx = \frac{r^2}{2}\left(\left(\frac{\pi}{2}\right) - \left(\arcsin(\cos\theta) + \sin\theta\cos\theta \right) \right)$$

$$0 < \theta < \frac{\pi}{2} \Rightarrow 0 < \frac{\pi}{2} - \theta < \frac{\pi}{2}$$

$$\cos(\theta) = \sin\left(\frac{\pi}{2} - \theta\right)$$

$$\Rightarrow \arcsin(\cos\theta) = \frac{\pi}{2} - \theta$$

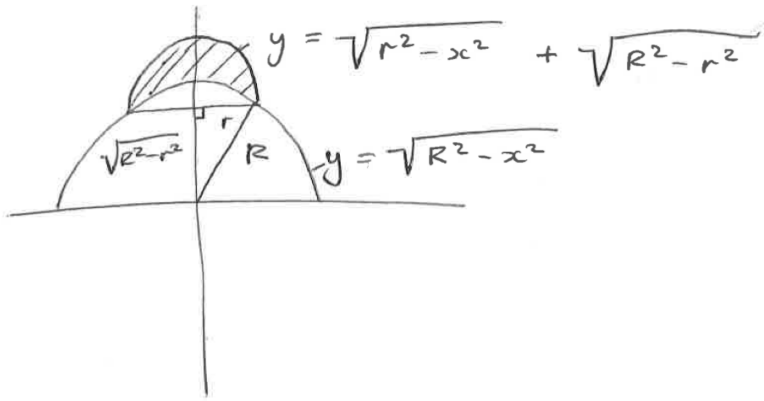
$$\Rightarrow \int_{r\cos\theta}^{r} \sqrt{r^2 - x^2}\, dx = \frac{r^2}{2}\left(\theta - \sin\theta\cos\theta \right)$$

$$\Rightarrow \text{Area} = \cancel{\frac{1}{2} r^2 \cos\theta \sin\theta} + \frac{r^2}{2}\left(\theta - \cancel{\sin\theta\cos\theta}\right)$$

$$= \frac{1}{2} r^2 \theta .$$

Q43 (Hard)

$0 < r < R$



$$\Rightarrow \quad \text{Area}(\equiv) = \int_{-r}^{r} \left(\sqrt{r^2 - x^2} + \sqrt{R^2 - r^2}\right) - \sqrt{R^2 - x^2}\, dx$$

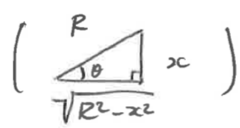
$$= 2r\sqrt{R^2 - r^2} + \int_{-r}^{r} \sqrt{r^2 - x^2}\, dx - \int_{-r}^{r} \sqrt{R^2 - x^2}\, dx$$

area of semi-circle radius r

$$= 2r\sqrt{R^2 - r^2} + \frac{1}{2}\pi r^2 - \int_{-r}^{r} \sqrt{R^2 - x^2}\, dx$$

$$\Rightarrow \quad \int \sqrt{R^2 - x^2}\, dx = R^2 \int \cos^2\theta\, d\theta = \frac{R^2}{2} \int (1 + \cos 2\theta)\, d\theta$$

$$= \frac{R^2}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + C$$



$$= \frac{R^2}{2}(\theta + \sin\theta\cos\theta) = \frac{R^2}{2}\left(\arcsin\left(\frac{x}{R}\right) + \frac{x\sqrt{R^2 - x^2}}{R^2}\right)$$

$$\Rightarrow \quad \int_{-r}^{r} \sqrt{R^2 - x^2}\, dx = \frac{R^2}{2}\left(\arcsin\left(\frac{r}{R}\right) + \frac{r\sqrt{R^2 - r^2}}{R^2}\right) - \frac{R^2}{2}\left(\arcsin\left(\frac{-r}{R}\right) - \frac{r\sqrt{R^2 - r^2}}{R^2}\right)$$

$$= R^2 \arcsin\left(\frac{r}{R}\right) + r\sqrt{R^2 - r^2}$$

$$\Rightarrow \quad \text{Area}(\equiv) = \frac{1}{2}\pi r^2 + r\sqrt{R^2 - r^2} - R^2 \arcsin\left(\frac{r}{R}\right).$$