Series

Let $\{a_n\}$ be a sequence. Does it make sense to talk about the infinite sum

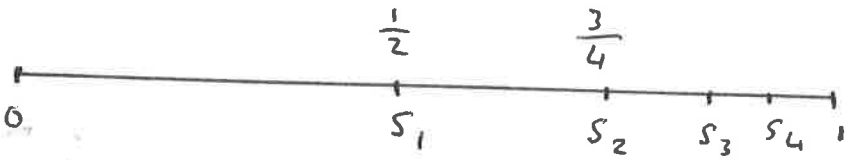
$$a_1 + a_2 + a_3 + \dots = \sum_{n=1}^{\infty} a_n \quad ?$$

Example

1/ Let $a_n = \frac{1}{2^n}$, i.e. $\{a_n\} = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots\}$

Let $s_n = \frac{1}{2} + \frac{1}{4} + \dots + \frac{1}{2^n}$

What happens as $n \to \infty$



As $n \to \infty$ we see that $s_n \to 1$. So even though it doesn't make sense to literally add together infinitely many numbers, we could define

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots = 1$$

2) Let $\{a_n\}_{n=1}^{\infty}$ be the sequence $a_n = n$, i.e. $\{1, 2, 3, 4, \ldots\}$. If we do the same thing here then

$$S_n = 1 + 2 + 3 + \cdots + n.$$

Clearly $S_n \geq n$ for all $n \geq 1$, hence $\lim_{n \to \infty} \{S_n\} = \infty$. So $\sum_{n=1}^{\infty} a_n$ doesn't have any meaning as a number in this case.

These observations inspire the following definition.

<u>Definition</u>

Let $\{a_n\}_{n=1}^{\infty}$ be a sequence. We define the <u>n^{th} partial sum</u> to be

$$S_n = \sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$$

If the sequence $\{S_n\}_{n=1}^{\infty}$ is convergent, with limit S, then we say the <u>infinite series</u> $\sum_{n=1}^{\infty} a_n$ is convergent and write $\sum_{n=1}^{\infty} a_n = S$.

If $\{S_n\}$ is <u>divergent</u> $\Rightarrow \sum_{n=1}^{\infty} a_n$ <u>divergent</u>.

For example : $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$ (convergent)

$\sum_{n=1}^{\infty} n$ divergent.

Core example : <u>Geometric Series</u>

a, r - real numbers , $a \neq 0$, $a_n = ar^{n-1}$

$\{a_n\} = \{a, ar, ar^2, ar^3, \ldots\}$

$S_n = a + ar + ar^2 + \ldots + ar^{n-1} \Rightarrow$

$rS_n = ar + ar^2 + \ldots + ar^{n-1} + ar^n \Rightarrow$

$S_n - rS_n = a - ar^n \Rightarrow (1-r)S_n = a(1-r^n)$

$r \neq 1 \Rightarrow S_n = \frac{a(1-r^n)}{1-r}$

<u>Case 1</u> $|r| < 1 \Rightarrow r^n \to 0$ as $n \to \infty \Rightarrow$

$\lim_{n\to\infty} \{S_n\} = \frac{a}{1-r} \Rightarrow \sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}$ (convergent)

<u>Case 2</u> $|r| > 1 \Rightarrow \{r^n\}$ divergent $\Rightarrow \{S_n\}$ divergent

$\Rightarrow \sum_{n=1}^{\infty} ar^{n-1}$ divergent.

Case 3 $r = -1 \Rightarrow \{s_n\} = \{a, 0, a, 0, a \ldots\}$

As $a \neq 0$ $\{s_n\}$ divergent $\Rightarrow \sum_{n=1}^{\infty} ar^{n-1}$ divergent

Case 4 $r = 1 \Rightarrow \{a_n\} = \{a, a, a, \ldots\} \Rightarrow$

$s_n = na \Rightarrow \{s_n\}$ divergent $\Rightarrow \sum_{n=1}^{\infty} ar^{n-1}$ divergent

Conclusion

$(a \neq 0)$

$$\sum_{n=1}^{\infty} ar^{n-1} = \begin{cases} \dfrac{a}{1-r} & \text{if } |r| < 1 \\ \text{divergent} & |r| \geq 1 \end{cases}$$

Fact: $\sum_{n=1}^{\infty} a_n$ convergent $\Rightarrow \lim_{n \to \infty} \{a_n\} = 0$

Warning: $\lim_{n \to \infty} \{a_n\} = 0 \not\Rightarrow \sum_{n=1}^{\infty} a_n$ convergent.

Example (Harmonic Series) $a_n = \frac{1}{n}$

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

$$s_2 = 1 + \frac{1}{2}$$

$$s_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{2}{2}$$

$$s_8 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)$$

$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) = 1 + \frac{3}{2}$$

$$s_{16} > 1 + \frac{4}{2}$$

$$s_{32} > 1 + \frac{5}{2}$$

$$\vdots$$

$$s_{2^n} > 1 + \frac{n}{2}$$

$1 + \frac{n}{2} \to \infty$ as $n \to \infty$ $\Rightarrow$ $S_{2^n} \to \infty$

as $n \to \infty$ $\Rightarrow$ $\{S_n\}$ divergent $\Rightarrow$ $\sum_{n=1}^{\infty} \frac{1}{n}$ divergent

Note however that $\lim_{n \to \infty} \left\{ \frac{1}{n} \right\} = 0$.

<u>Important</u>: It is not easy to spot if $\sum_{n=1}^{\infty} a_n$ is convergent just by looking at $\{a_n\}$.