DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.

Formulae

$$\int \tan(x)\,dx = \ln|\sec(x)| + C \qquad \int \sec(x)\,dx = \ln|\sec(x)+\tan(x)| + C$$

$$\int \frac{1}{1+x^2}dx = \arctan(x) + C \qquad \int \frac{1}{\sqrt{1-x^2}}dx = \arcsin(x) + C$$

$$\frac{d\tan(x)}{dx} = \sec^2(x) \qquad \frac{d\sec(x)}{dx} = \tan(x)\sec(x)$$

$$1 = \sin^2(x) + \cos^2(x) \qquad 1 + \tan^2(x) = \sec^2(x)$$

$$\cos^2(x) = \frac{1+\cos(2x)}{2} \qquad \sin^2(x) = \frac{1-\cos(2x)}{2}$$

$$e^x = 1 + x + \tfrac{x^2}{2} + \tfrac{x^3}{6} + \cdots = \textstyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin x = x - \tfrac{x^3}{3!} + \tfrac{x^5}{5!} - \tfrac{x^7}{7!} + \tfrac{x^9}{9!} - \cdots = \textstyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

$$\cos x = 1 - \tfrac{x^2}{2!} + \tfrac{x^4}{4!} - \tfrac{x^6}{6!} + \tfrac{x^8}{8!} - \cdots = \textstyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$$

$$\arctan x = x - \tfrac{x^3}{3} + \tfrac{x^5}{5} - \tfrac{x^7}{7} + \tfrac{x^9}{9} - \cdots = \textstyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$\ln(1+x) = x - \tfrac{x^2}{2} + \tfrac{x^3}{3} - \tfrac{x^4}{4} + \tfrac{x^5}{5} - \cdots = \textstyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

$$\lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n = e$$

CALCULATORS ARE NOT PERMITTED

Name and section: 1B Midterm 2 (001) Master

GSI's name: _____

Blank Page

Math 1B Second Midterm Exam (LEC 001) 9:10 am to 10am

This exam consists of 5 questions. Answer the questions in the spaces provided.

1. Determine if the following series converge or diverge. If convergent you do not need to give the sum. Carefully justify your answers.

   (a) (10 points)

$$\sum_{n=1}^{\infty} \cos\left(\frac{n}{n^2+1}\right)$$

Solution:

$\lim_{n\to\infty} \left\{\frac{n}{n^2+1}\right\} = 0 \Rightarrow \lim_{n\to\infty} \left\{\cos\left(\frac{n}{n^2+1}\right)\right\} = 1 \neq 0$

$\Rightarrow$ Divergent by divergence test.

   (b) (15 points)

$$\sum_{n=1}^{\infty} \frac{3^n}{7^n - 6^n}$$

Solution:

$a_n = \frac{3^n}{7^n - 6^n}$, $b_n = \frac{3^n}{7^n} = \left(\frac{3}{7}\right)^n$

$\frac{a_n}{b_n} = \frac{21^n}{21^n - 18^n} = \frac{1}{1 - \left(\frac{18}{21}\right)^n} \to 1$ as $n \to \infty$

$\sum_{n=1}^{\infty} \left(\frac{3}{7}\right)^n$ conv. (geo series $r = \frac{3}{7}$) $\Rightarrow$ (Limit comparison) $\sum_{n=1}^{\infty} \frac{3^n}{7^n - 6^n}$ conv.

2. (20 points) Determine if the following series is convergent or divergent. If convergent you do not need to determine the sum.

$$\sum_{n=1}^{\infty} \frac{3^n \cdot n!}{1 \cdot 3 \cdot 5 \cdots (2n-1)}$$

Solution:

$$a_n = \frac{3^n n!}{1 \cdot 3 \cdot 5 \cdots (2n-1)} \Rightarrow \left| \frac{a_{n+1}}{a_n} \right| = \frac{3 \cdot (n+1)}{2n+1}$$

$$\Rightarrow \lim_{n \to \infty} \left\{ \left| \frac{a_{n+1}}{a_n} \right| \right\} = \frac{3}{2} > 1$$

$\Rightarrow$ Divergent by ratio test.

3. (30 points) Using the integral test, determine whether the following series is absolutely convergent, conditionally convergent or divergent. If convergent you do not need to determine the sum.

$$\sum_{n=1}^{\infty}(-1)^n\frac{\sin(1/n^2)}{n^3}$$

Solution:

$f(x) = \dfrac{\sin(1/x^2)}{x^3}$

1/ $f(x) > 0$ on $[1, \infty)$

2/ $f'(x) = \dfrac{\frac{-2}{x^3}\cos(1/x^2) - 3x^2\sin(1/x^3)}{x^6} \leq 0$

on $[1, \infty)$

3/ $x \neq 0$ on $[1, \infty)$ $\Rightarrow$ $f(x)$ continuous.

$u = 1/x^2 \Rightarrow \dfrac{du}{dx} = \dfrac{-2}{x^3} \Rightarrow dx = \dfrac{x^3\,du}{-2}$

$\Rightarrow \int f(x)\,dx = -\frac{1}{2}\int \sin(u)\,du = \frac{1}{2}\cos(1/x^2) + C$

$\Rightarrow \int_1^{\infty} f(x)\,dx = \lim_{t\to\infty} \frac{1}{2}\cos(1/t^2) - \frac{1}{2}\cos(1)$

$= \frac{1}{2} - \frac{1}{2}\cos(1)$ $\Rightarrow$ convergent $\Rightarrow$ Series absolutely convergent by Integral test.

4. (25 points) Determine the domain of the function $f(x)$ given by the power series

$$\sum_{n=1}^{\infty}(-1)^n\frac{(3x-1)^n}{\sqrt{n}}.$$

Solution:

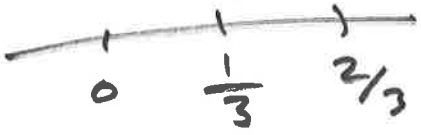
$a_n = (-1)^n \dfrac{(3x-1)^n}{\sqrt{n}} \Rightarrow \left|\dfrac{a_{n+1}}{a_n}\right| = \sqrt{\dfrac{n+1}{n}} \cdot |3x-1|$

$\Rightarrow \lim_{n\to\infty}\left\{\left|\dfrac{a_{n+1}}{a_n}\right|\right\} = |3x-1|$

$|3x-1| < 1$ (ie $|x-\frac{1}{3}| < \frac{1}{3}$) $\Rightarrow$ conv.

$|3x-1| > 1$ (ie $|x-\frac{1}{3}| > \frac{1}{3}$) $\Rightarrow$ div.

$\Rightarrow R = \frac{1}{3}$



$x=0$ $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ div $(p=\frac{1}{2}<1)$

$x=2/3$ $\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$ conv by A.S.T.

$\Rightarrow$ Domain $= (0, \frac{2}{3}]$

5. (a) (15 points) Calculate the Maclaurin series of the function

$$f(x) = x^2 \ln(1 - 2x).$$

Make sure to write a general term of the series.

Solution:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} \cdots \quad |x| < 1$$

$$\Rightarrow \ln(1-2x) = -2x - \frac{(-2x)^2}{2} + \frac{(-2x)^3}{3} \cdots \quad |-2x| < 1 \ (\text{ie } |x| < \tfrac{1}{2})$$

$$\Rightarrow x^2 \ln(1-2x) = -2x^3 - \frac{2^2 x^4}{2} + \frac{-2^3 x^5}{3} \cdots$$

$$= -\sum_{n=1}^{\infty} \frac{2^n x^{n+2}}{n} \quad \text{for all } |x| < \frac{1}{2}$$

$$\Rightarrow \text{Maclaurin Series is } \sum_{n=1}^{\infty} -\frac{2^n x^{n+2}}{n}.$$

(b) (10 points) Using part (a), or otherwise, calculate $f^{(100)}(0)$. You do not need to simplify your answer.

Solution:

$$\frac{f^{(100)}(0)}{100!} = \frac{-2^{98}}{98} \Rightarrow f^{(100)}(0) = \frac{-2^{98} \cdot 100!}{98}$$

END OF EXAM

Blank Page

Blank Page

Blank Page