This exam consists of 5 questions. Answer the questions in the spaces provided.

Name and section: ______________________

GSI's name: ______________________

1. Determine if the following sequences converge or diverge. Carefully justify your answer.

(a) (10 points)

$$\left\{\frac{\cos(n)}{\sqrt{n}}\right\}_{n=1}^{\infty}$$

Solution:

Observe that $-1 \leq \cos(n) \leq 1$ for all $n \geq 1$

Hence $\frac{-1}{\sqrt{n}} \leq \frac{\cos(n)}{\sqrt{n}} \leq \frac{1}{\sqrt{n}}$ for all $n \geq 1$

$\sqrt{n} \to \infty$ as $n \to \infty \Rightarrow \frac{-1}{\sqrt{n}}, \frac{1}{\sqrt{n}} \to 0$ as $n \to \infty$

Convergent by Squeeze theorem.

(b) (10 points)

$$\left\{n\sin\left(\frac{1}{n}\right)\right\}_{n=1}^{\infty}$$

Solution:

Let $f(x) = x \sin\left(\frac{1}{x}\right) = \frac{\sin\left(\frac{1}{x}\right)}{\left(\frac{1}{x}\right)}$

$\lim\limits_{n \to \infty} \left\{ n \sin\left(\frac{1}{n}\right)\right\} = \lim\limits_{x \to \infty} f(x) \underset{\text{l'Hopital}}{=} \lim\limits_{x \to \infty} \frac{\frac{-1}{x^2}\cos\left(\frac{1}{x}\right)}{\frac{-1}{x^2}}$

$= \lim\limits_{x \to \infty} \cos\left(\frac{1}{x}\right) = \cos(0) = 1.$

Hence sequence convergent.

2. Determine whether the following series are convergent or divergent. If convergent determine the sum.

   (a) (10 points)

$$\sum_{n=1}^{\infty} \ln\left(\frac{n}{n+1}\right)$$

   (Hint: Try to explicitly determine the partial sums)

   Solution:

Observe that $\ln\left(\frac{n}{n+1}\right) = \ln(n) - \ln(n+1)$.

$s_1 = \ln(1) - \ln(2) = -\ln(2)$

$s_2 = \ln(1) - \ln(2) + \ln(2) - \ln(3) = \ln(1) - \ln(3) = -\ln(3)$

$\vdots$ Telescoping sum

$s_n = \ln(1) - \ln(n+1) = -\ln(n+1)$.

$-\ln(n+1) \to -\infty$ as $n \to \infty$, hence series divergent.

   (b) (10 points)

$$\sum_{n=1}^{\infty} \frac{\sqrt{4n^2+2n+1}}{4n+6}$$

   Solution:

$$\frac{\sqrt{4n^2+2n+1}}{4n+6} = \frac{\sqrt{4+\frac{2}{n}+\frac{1}{n^2}}}{4+\frac{6}{n}}$$

$\frac{1}{n} \to 0$ as $n \to \infty$, hence

$$\frac{\sqrt{4+\frac{2}{n}+\frac{1}{n^2}}}{4+\frac{6}{n}} \to \frac{\sqrt{4}}{4} = \frac{1}{2} \neq 0 \quad \text{as } n \to \infty.$$

Hence divergent by divergence test.

3. (20 points) Determine whether the following series is convergent or divergent. If convergent you do not need to determine the sum.

$$\sum_{n=1}^{\infty} \frac{\sin(n)}{7^n - 3^n}$$

Solution:

First check absolute convergence.

$\left|\frac{\sin(n)}{7^n-3^n}\right| < \frac{1}{7^n-3^n}$, hence consider $\sum_{n=1}^{\infty} \frac{1}{7^n-3^n}$

Do limit comparison test with $\sum_{n=1}^{\infty} \frac{1}{7^n}$

$$\frac{\left(\frac{1}{7^n-3^n}\right)}{\frac{1}{7^n}} = \frac{7^n}{7^n-3^n} = \frac{1}{1-\left(\frac{3}{7}\right)^n} \longrightarrow 1 \text{ as}$$

$n \to \infty$, because $\frac{3}{7} < 1$. Because $\frac{1}{7^n} = \left(\frac{1}{7}\right)^n$ and $\frac{1}{7} < 1$, this is a <u>convergent</u> geometric series.

Hence by limit comparison test $\sum_{n=1}^{\infty} \frac{1}{7^n-3^n}$ convergent.

By comparison test $\sum_{n=1}^{\infty} \frac{\sin(n)}{7^n-3^n}$ <u>absolutely convergent.</u>

We deduce the series is <u>convergent</u>

4. (20 points) Determine whether the following series is absolutely convergent, conditionally convergent or divergent. If convergent you do not need to determine the sum.

$$\sum_{n=1}^{\infty}(-1)^{n-1}ne^{-n^2} = e^{-1} - 2e^{-4} + 3e^{-9} + \cdots$$

Solution:

First check absolute convergence.

Let $f(x) = x e^{-x^2} \Rightarrow f'(x) = e^{-x^2} - 2x^2 e^{-x^2}$

$= (1 - 2x^2) e^{-x^2} < 0$

for all $x \geq 1$. Thus we may check absolute convergence using integral test.

$$\int_1^\infty x e^{-x^2} dx = \lim_{t \to \infty} \left[\frac{-1}{2} e^{-x^2} \right]_1^t = \lim_{t \to \infty} \left(\frac{1}{2e} - \frac{1}{2} e^{-t^2} \right)$$

$$= \frac{1}{2e}.$$

Hence $\sum_{n=1}^{\infty} (-1)^{n-1} n e^{-n^2}$ is absolutely convergent.

This could also be done using either the root or ratio tests.

5. (20 points) Determine whether the following series is convergent or divergent. If convergent you do not need to determine the sum.

$$\sum_{n=1}^{\infty} \frac{n^n}{(2n)!}$$

Solution:

Try ratio test. Let $a_n = \frac{n^n}{(2n)!} \Rightarrow$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)^{n+1}}{n^n} \cdot \frac{1}{(2n+1)(2n+2)} = \frac{(n+1)^n}{n} \cdot \frac{n+1}{(2n+1)(2n+2)}$$

$$= \frac{(n+1)^n}{n^n} \cdot \frac{1}{2(2n+1)}$$

But $\left(\frac{n+1}{n}\right)^n = \left(1+\frac{1}{n}\right)^n \rightarrow e$ as $n \rightarrow \infty$.

$\frac{1}{2(2n+1)} \rightarrow 0$ as $n \rightarrow \infty$, hence

$\left|\frac{a_{n+1}}{a_n}\right| \rightarrow 0$ as $n \rightarrow \infty$. Hence convergent by ratio test.

We could also show this by observing $\frac{n^n}{(n+1) \dots (2n)}$

$= \frac{n \times n \dots \times n}{(n+1)\times(n+2) \dots \times(2n)} < 1 \Rightarrow \frac{n^n}{2n!} < \frac{1}{n!}$. We could then prove $\sum_{n=1}^{\infty} \frac{1}{n!}$ convergent by ratio test.

END OF EXAM

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