MATH 1B MIDTERM 1 (PRACTICE 1)
PROFESSOR PAULIN

DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.

CALCULATORS ARE NOT PERMITTED

THIS EXAM WILL BE ELECTRONICALLY SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED. YOU MAY WRITE SOLUTIONS ON THE BLANK PAGE AT THE BACK BUT BE SURE TO CLEARLY LABEL THEM

Formulae

$$\int \tan(x)\ dx = \ln|\sec(x)| + C \qquad \int \sec(x)\ dx = \ln|\sec(x)+\tan(x)| + C$$

$$\int \frac{1}{1+x^2}dx = \arctan(x) + C \qquad \int \frac{1}{\sqrt{1-x^2}}dx = \arcsin(x) + C$$

$$\frac{d\tan(x)}{dx} = \sec^2(x) \qquad \frac{d\sec(x)}{dx} = \tan(x)\sec(x)$$

$$1 = \sin^2(x) + \cos^2(x) \qquad 1 + \tan^2(x) = \sec^2(x)$$

$$\cos^2(x) = \frac{1+\cos(2x)}{2} \qquad \sin^2(x) = \frac{1-\cos(2x)}{2}$$

$$|E_{Mid_n}| \leq \frac{K(b-a)^3}{24n^2} \qquad |E_{S_n}| \leq \frac{K(b-a)^5}{180n^4}$$

Name: ____

Student ID: ____

GSI's name: ____

This exam consists of 5 questions. Answer the questions in the spaces provided.

1. Compute the following integrals:

   (a) (10 points)

$$\int x \sec^2(x)\, dx$$

Solution:

$$\left.\begin{aligned} f(x) &= x, & g'(x) &= \sec^2(x) \\ f'(x) &= 1, & g(x) &= \tan(x) \end{aligned}\right\} \Rightarrow \int x\sec^2(x)\,dx = x\tan(x) - \int \tan(x)\,dx$$

$$\Rightarrow \int x\sec^2(x)\,dx = x\tan(x) - \ln|\sec(x)| + C$$

   (b) (15 points)

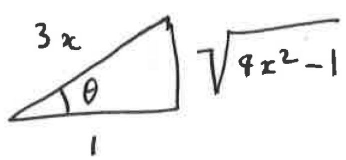
$$\int \frac{1}{\sqrt{9x^2-1}}\, dx$$

Solution:

$$x = \frac{1}{3}\sec\theta \Rightarrow \frac{dx}{d\theta} = \frac{1}{3}\tan\theta\sec\theta \Rightarrow dx = \frac{1}{3}\tan\theta\sec\theta\, d\theta$$

$$\Rightarrow \int \frac{1}{\sqrt{9x^2-1}}\, dx = \frac{1}{3}\int \frac{\tan\theta\sec\theta}{\tan\theta}\, d\theta = \frac{1}{3}\int \sec\theta\, d\theta$$

$$= \frac{1}{3}\ln|\tan\theta + \sec\theta| + C = \frac{1}{3}\ln\left|\sqrt{9x^2-1} + 3x\right| + C$$

(

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2. (a) (15 points) Express the following rational function

$$\frac{2x^2 + 4x + 1}{x^3 + 2x^2 + x}$$

as a sum of partial fractions.

Solution:

$$\frac{2x^2+4x+1}{x^3+2x^2+x} = \frac{2x^2+4x+1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$$

$$= \frac{A(x+1)^2 + Bx(x+1) + Cx}{x(x+1)^2} = \frac{(A+B)x^2 + (2A+B+C)x + A}{x(x+1)^2}$$

$$\Rightarrow \begin{aligned} A + B &= 2 \\ 2A + B + C &= 4 \\ A &= 1 \end{aligned} \Rightarrow \begin{aligned} A &= 1 \\ B &= 1 \\ C &= 1 \end{aligned} \Rightarrow \frac{2x^2+4x+1}{x^3+2x^2+x} = \frac{1}{x} + \frac{1}{x+1} + \frac{1}{(x+1)^2}$$

(b) (10 points) Hence evaluate the integral

$$\int \frac{2x^2 + 4x + 1}{x^3 + 2x^2 + x}\, dx$$

Solution:

$$\int \frac{2x^2+4x+1}{x^3+2x^2+x}\,dx = \int \frac{1}{x}\,dx + \int \frac{1}{x+1}\,dx + \int \frac{1}{(x+1)^2}\,dx$$

$$= \ln|x| + \ln|x+1| - \frac{1}{x+1} + C$$

3. (25 points) Find the arc length of the curve

$$y = \frac{(x+1)^3}{3} + \frac{1}{4x+4}$$

between $x = 0$ and $x = 1$.

Solution:

$$f(x) = \frac{(x+1)^3}{3} + \frac{1}{4(x+1)} \implies f'(x) = (x+1)^2 - \frac{1}{4(x+1)^2}$$

$$\implies 1 + f'(x)^2 = 1 + (x+1)^4 - \frac{1}{2} + \frac{1}{16(x+1)^4}$$

$$= (x+1)^4 + \frac{1}{2} + \frac{1}{16(x+1)^4}$$

$$= \left((x+1)^2 + \frac{1}{4(x+1)^2}\right)^2$$

$$\implies \text{Arc Length} = \int_0^1 (x+1)^2 + \frac{1}{4(x+1)^2}\, dx$$

$$= \frac{(x+1)^3}{3} - \frac{1}{4(x+1)} \Big|_0^1 = \frac{8}{3} - \frac{1}{8} - \frac{1}{3} + \frac{1}{4}$$

$$= \frac{7}{3} + \frac{1}{8} = \frac{56+3}{24} = \frac{59}{24}$$

4. Determine if the following improper integrals are convergent or divergent. Justify your answers.

   (a) (10 points)
$$\int_0^\infty \frac{4+\sin(x)}{e^{3x}}dx$$

   Solution:

$$0 < 4+\sin(x) \le 5 \quad \Rightarrow \quad 0 < \frac{4+\sin(x)}{e^{3x}} \le \frac{5}{e^{3x}} \text{ on } [0,\infty)$$

$$\int_0^\infty \frac{5}{e^{3x}}dx = \lim_{t\to\infty} \int_0^t \frac{5}{e^{3x}}dx = \lim_{t\to\infty} \frac{5}{-3}e^{-3x}\Big|_0^t = \lim_{t\to\infty} \frac{5}{-3}(e^{-3t}-1) = \frac{5}{3}$$

$$\Rightarrow \int_0^\infty \frac{5}{e^{3x}}dx \text{ convergent} \Rightarrow \int_0^\infty \frac{4+\sin(x)}{e^{3x}}dx \ \underline{\text{convergent}}$$

   (b) (15 points)
$$\int_0^5 \frac{2x}{x^2-1}dx$$

$$\int_0^5 \frac{2x}{x^2-1}dx = \int_0^1 \frac{2x}{x^2-1}dx + \int_1^5 \frac{2x}{x^2-1}dx$$

$$\int \frac{2x}{x^2-1}dx = \ln|x^2-1| + C \Rightarrow$$

$$\int_1^5 \frac{2x}{x^2-1}dx = \lim_{t\to 1^+} \ln|x^2-1|\Big|_t^5 = \lim_{t\to 1^+}(\ln(24) - \ln|t^2-1|)$$

$$= \infty \Rightarrow \int_0^5 \frac{2x}{x^2-1}dx \ \underline{\text{divergent}}.$$

5. (a) (15 points) For n a positive integer, let Mid_n be the midpoint approximation of the definite integral

$$\int_0^1 e^{2x^2}\,dx.$$

How large do we need to choose n to be to guarantee that the estimate is within 0.01 of the true value? You do not need to give an exact answer, just a carefully justified bound.

Solution:

$f(x) = e^{2x^2} \Rightarrow f'(x) = 4x\,e^{2x^2} \Rightarrow f''(x) = 4e^{2x^2} + 16x^2e^{2x^2}$

$= (16x^2+4)e^{2x^2}$. This is increasing on $[0,1]$, so let

$K = (16\cdot 1^2+4)e^{2\cdot 1^2} = 20e^2$. Need to choose $n \geq 1$

such that $\dfrac{20e^2(1-0)^3}{24n^2} \leq 0.01$

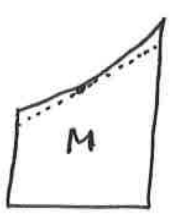
$\Rightarrow n^2 \geq \dfrac{2000e^2}{24} \Rightarrow n \geq \sqrt[2]{\dfrac{2000e^2}{24}} = \sqrt[2]{\dfrac{2000}{24}}\cdot e$

(b) (10 points) Is this approximation an overestimate or an underestimate? Be sure to justify your answer.

Solution:

$f''(x) > 0$ on $[0,1]$ $\Rightarrow$ $y = f(x)$ concave up on $[0,1]$

$\Rightarrow$ Mid_n is an underestimate.



END OF EXAM