MATH 1B MIDTERM 1 (002)
PROFESSOR PAULIN

DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.

CALCULATORS ARE NOT PERMITTED

THIS EXAM WILL BE ELECTRONICALLY SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED. YOU MAY WRITE SOLUTIONS ON THE BLANK PAGE AT THE BACK BUT BE SURE TO CLEARLY LABEL THEM

Formulae

$$\int \tan(x)\ dx = \ln|\sec(x)| + C \qquad \int \sec(x)\ dx = \ln|\sec(x) + \tan(x)| + C$$

$$\int \frac{1}{1+x^2}dx = \arctan(x) + C \qquad \int \frac{1}{\sqrt{1-x^2}}dx = \arcsin(x) + C$$

$$\frac{d\tan(x)}{dx} = \sec^2(x) \qquad \frac{d\sec(x)}{dx} = \tan(x)\sec(x)$$

$$1 = \sin^2(x) + \cos^2(x) \qquad 1 + \tan^2(x) = \sec^2(x)$$

$$\cos^2(x) = \frac{1+\cos(2x)}{2} \qquad \sin^2(x) = \frac{1-\cos(2x)}{2}$$

$$|E_{T_n}| \leq \frac{K(b-a)^3}{12n^2} \qquad |E_{S_n}| \leq \frac{K(b-a)^5}{180n^4}$$

Name: ______________________________

Student ID: ______________________________

GSI's name: ______________________________

This exam consists of 5 questions. Answer the questions in the spaces provided.

1. Compute the following integrals:

   (a) (10 points)

$$\int x\sin(x)\,dx$$

Solution:

$$\begin{array}{ccc} & & -\cos(x) \\ & & | \\ x & — & \sin(x) \\ | & & \\ 1 & & \end{array}$$

$$\int x\sin(x)\,dx = -x\cos(x) + \int \cos(x)\,dx$$
$$= -x\cos(x) + \sin(x) + C$$

   (b) (15 points)

$$\int \frac{\sqrt{x^2-1}}{x^4}\,dx$$

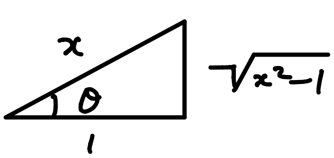
Solution:

$$x = \sec\theta \Rightarrow \frac{dx}{d\theta} = \tan\theta\sec\theta$$

$$\Rightarrow \int \frac{\sqrt{x^2-1}}{x^4}\,dx = \int \frac{\tan\theta}{\sec^4\theta}\tan\theta\sec\theta\,d\theta = \int \sin^2\theta\cos\theta\,d\theta$$

$$u = \sin\theta \Rightarrow \frac{du}{d\theta} = \cos\theta \Rightarrow d\theta = \frac{du}{\cos\theta} \Rightarrow \int \sin^2\theta\cos\theta\,d\theta = \int u^2\,du$$

$$= \frac{1}{3}u^3 + C = \frac{1}{3}(\sin\theta)^3 + C = \frac{1}{3}\left(\frac{\sqrt{x^2-1}}{x}\right)^3 + C$$



$\longleftarrow$ $\sec\theta = \frac{x}{1}$

2. (a) (15 points) Express the following rational function

$$\frac{3x^3+2x^2+x+1}{x^4+x^2}$$

as a sum of partial fractions.

Solution:

$$\frac{3x^3+2x^2+x+1}{x^4+x^2} = \frac{3x^3+2x^2+x+1}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$$

$$= \frac{Ax(x^2+1) + B(x^2+1) + (Cx+D)x^2}{x^4+x^2}$$

$$= \frac{(A+C)x^3 + (B+D)x^2 + Ax + B}{x^4+x^2}$$

$$\Rightarrow \begin{array}{l} A+C = 3 \\ B+D = 2 \\ A = 1 \\ B = 1 \end{array} \Rightarrow \begin{array}{l} C = 2 \\ D = 1 \end{array} \Rightarrow \frac{3x^2+2x^2+x+1}{x^4+x^2} = \frac{1}{x} + \frac{1}{x^2} + \frac{2x+1}{x^2+1}$$

(b) (10 points) Hence evaluate the integral

$$\int \frac{3x^3+2x^2+x+1}{x^4+x^2}\,dx$$

Solution:

$$\int \frac{3x^3+2x^2+x+1}{x^4+x}\,dx = \ln|x| - \frac{1}{x} + \ln|x^2+1| + \arctan(x) + C$$

3. (25 points) Find the arc length of the curve

$$y = \frac{x^2}{4} - \ln(\sqrt{x})$$

between $x = 1$ and $x = 2$.

Solution:

$$f(x) = \frac{x^2}{4} - \ln(\sqrt{x}) = \frac{x^2}{4} - \frac{1}{2}\ln(x)$$

$$\Rightarrow f'(x) = \frac{x}{2} - \frac{1}{2x}$$

$$\Rightarrow 1 + f'(x)^2 = \left(\frac{x}{2}\right)^2 + \frac{1}{2} + \left(\frac{1}{2x}\right)^2 = \left(\frac{x}{2} + \frac{1}{2x}\right)^2$$

$$\Rightarrow \text{Arc Length} = \int_1^2 \frac{x}{2} + \frac{1}{2x}\,dx = \frac{x^2}{4} + \frac{1}{2}\ln|x| \Big|_1^2$$

$$= \left(1 + \frac{1}{2}\ln(2)\right) - \left(\frac{1}{4}\right)$$

$$= \frac{3}{4} + \frac{1}{2}\ln(2)$$

4. Determine if the following improper integrals are convergent or divergent. Justify your answers.

(a) (10 points)

$$\int_{-\infty}^{-1} \frac{\sin(x^2)+2}{x^2}dx$$

Solution:

$$0 \leq \sin(x^2)+2 \leq 3 \Rightarrow 0 \leq \frac{\sin(x^2)+2}{x^2} \leq \frac{3}{x^2} \text{ on } (-\infty,-1]$$

$$\int_{-\infty}^{-1} \frac{1}{x^2}dx = \lim_{t\to-\infty} \int_t^{-1} \frac{1}{x^2}dx = \lim_{t\to-\infty} \left.\frac{-1}{x}\right|_t^{-1}$$

$$= \lim_{t\to-\infty} 1 + \frac{1}{t} = 1$$

$$\int_{-\infty}^{-1} \frac{1}{x^2}dx \quad \underline{\text{conv}} \quad \Rightarrow \quad \int_{-\infty}^{-1} \frac{\sin(x^2)+2}{x^2}dx \quad \underline{\text{conv}}$$

(b) (15 points)

$$\int_0^1 \frac{\cos(1/x)}{x^2}dx$$

$$u = 1/x \Rightarrow \frac{du}{dx} = \frac{-1}{x^2} \Rightarrow dx = -x^2\,du$$

$$\Rightarrow \int \frac{\cos(\frac{1}{x})}{x^2}dx = \int -\cos(u)\,du = -\sin(u)+c = -\sin(1/x)+c$$

$$\Rightarrow \int_0^1 \frac{\cos(1/x)}{x^2}dx = \lim_{t\to0^+} \int_t^1 \frac{\cos(1/x)}{x^2}dx = \lim_{t\to0^+} \left.-\sin(1/x)\right|_t^1$$

$$= \lim_{t\to0^+} \sin(1/t) - \sin(1) \quad \text{DNE}$$

$$\Rightarrow \int_0^1 \frac{\cos(1/x)}{x^2}dx \quad \textit{Divergent}$$

5. (25 points) For n a positive integer, let T_n be the trapezoidal approximation of the definite integral

$$\int_{-1}^{1} x^4 - 6x^2 + x + \frac{23}{10}\,dx.$$

Is it possible that $T_{100} = 0.99$? Carefully justify your answer. Hint: What is the exact value of the integral?

Solution:

$$f(x) = x^4 - 6x^2 + x + \frac{23}{10}$$

$$\Rightarrow f'(x) = 4x^3 - 12x + 1 \Rightarrow f''(x) = 12x^2 - 12$$

on $[-1,1]$

$$|f''(x)| = |12x^2 - 12| \leq 12|x|^2 + 12 \leq 24$$

$$\Rightarrow |E_{T_{100}}| \leq \frac{24 \cdot 2^3}{12 \cdot 100^2} = \frac{2^4}{100^2} = 0.0016$$

$$\int_{-1}^{1} x^4 - 6x^2 + x + \frac{23}{10}\,dx = \frac{x^5}{5} - 2x^3 + \frac{1}{2}x^2 + \frac{23}{10}x \Big|_{-1}^{1}$$

$$= \frac{2}{5} - 4 + \frac{23}{5} = 1$$

$$T_{100} = 0.99 \Rightarrow |E_{T_{100}}| = 0.01 \not\leq 0.0016$$

$$\Rightarrow T_{100} \neq 0.99$$

END OF EXAM