MATH 1B FINAL (PRACTICE 2)
PROFESSOR PAULIN

DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.

CALCULATORS ARE NOT PERMITTED

THIS EXAM WILL BE ELECTRONICALLY SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED. YOU MAY WRITE SOLUTIONS ON THE BLANK PAGE AT THE BACK BUT BE SURE TO CLEARLY LABEL THEM

$$\int \tan(x)\,dx = \ln|\sec(x)| + C \qquad \int \sec(x)\,dx = \ln|\sec(x)+\tan(x)| + C$$

$$\cos^2(x) = \frac{1+\cos(2x)}{2} \qquad \sin^2(x) = \frac{1-\cos(2x)}{2}$$

$$|E_{Mid_n}| \leq \frac{K(b-a)^3}{24n^2} \qquad |E_{S_n}| \leq \frac{K(b-a)^5}{180n^4}$$

$$e^x = 1 + x + \tfrac{x^2}{2} + \tfrac{x^3}{6} + \cdots = \textstyle\sum_{n=0}^{\infty} \tfrac{x^n}{n!}$$

$$\sin x = x - \tfrac{x^3}{3!} + \tfrac{x^5}{5!} - \tfrac{x^7}{7!} + \tfrac{x^9}{9!} - \cdots = \textstyle\sum_{n=0}^{\infty}(-1)^n \tfrac{x^{2n+1}}{(2n+1)!}$$

$$\cos x = 1 - \tfrac{x^2}{2!} + \tfrac{x^4}{4!} - \tfrac{x^6}{6!} + \tfrac{x^8}{8!} - \cdots = \textstyle\sum_{n=0}^{\infty}(-1)^n \tfrac{x^{2n}}{(2n)!}$$

$$\arctan x = x - \tfrac{x^3}{3} + \tfrac{x^5}{5} - \tfrac{x^7}{7} + \tfrac{x^9}{9} - \cdots = \textstyle\sum_{n=0}^{\infty}(-1)^n \tfrac{x^{2n+1}}{2n+1}$$

$$\ln(1+x) = x - \tfrac{x^2}{2} + \tfrac{x^3}{3} - \tfrac{x^4}{4} + \tfrac{x^5}{5} - \cdots = \textstyle\sum_{n=1}^{\infty}(-1)^{n-1} \tfrac{x^n}{n}$$

$$(1+x)^k = 1 + kx + \tfrac{k(k-1)}{2!}x^2 + \tfrac{k(k-1)(k-2)}{3!}x^3 + \cdots = \textstyle\sum_{n=0}^{\infty} \binom{k}{n} x^n$$

$$\lim_{n\to\infty}\left(\tfrac{n+1}{n}\right)^n = e$$

Name: ______________________________

Student ID: ______________________________

GSI's name: ______________________________

This exam consists of 10 questions. Answer the questions in the spaces provided.

1. Compute the following integrals:

   (a) (10 points)

$$\int \arctan(x)dx$$

Solution:

$$f(x) = \arctan(x) \qquad g'(x) = 1$$

$$f'(x) = \frac{1}{1+x^2} \qquad g(x) = x$$

$$\Rightarrow \int \arctan(x)\,dx = x\arctan(x) - \int \frac{x}{1+x^2}\,dx$$

$$u = 1+x^2 \Rightarrow \frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x}$$

$$\Rightarrow \int \frac{x}{1+x^2}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C$$

$$= \frac{1}{2}\ln|1+x^2| + C$$

$$\Rightarrow \int \arctan(x)\,dx = x\arctan(x) - \frac{1}{2}\ln|1+x^2| + C$$

(b) (15 points)

$$\int \cos(\ln(x))\, dx$$

Hint: try a substitution first.

Solution:

Let $u = \ln(x) \Rightarrow \frac{du}{dx} = \frac{1}{x} \Rightarrow dx = x\,du$

$\Rightarrow \int \cos(\ln(x))\, dx = \int x \cos(u)\, du = \int e^u \cos(u)\, du$

Let $f(u) = \cos(u)$, $g'(u) = e^u$

$f'(u) = -\sin(u)$, $g(u) = e^u$

$\Rightarrow \int e^u \cos(u)\, du = e^u \cos(u) + \int e^u \sin(u)\, du$

Let $f(u) = \sin(u)$, $g'(u) = e^u$

$f'(u) = \cos(u)$, $g(u) = e^u$

$\Rightarrow \int e^u \cos(u)\, du = e^u \cos(u) + e^u \sin(u) - \int e^u \cos(u)\, du$

$\Rightarrow \int e^u \cos(u)\, du = \frac{1}{2}(\cos(u) + \sin(u))e^u + C$

$\Rightarrow \int \cos(\ln(x))\, dx = \frac{1}{2}(\cos(\ln(x)) + \sin(\ln(x)))\,x + C.$

2. (25 points) Determine if the following series are convergent or divergent. You do not need to show your working.

(a)

$$\sum_{n=2}^{\infty}(\ln(n+1)-\ln(n))$$

Solution:

Divergent

(b)

$$\sum_{n=1}^{\infty}\frac{n^6-1}{n^7+2}$$

Solution:

Divergent

(c)

$$\sum_{n=1}^{\infty}\frac{\sin(2n)}{2^n}$$

Solution:

Convergent

(d)

$$\sum_{n=1}^{\infty}\left(\frac{n}{n+1}\right)^{n^2}$$

Solution:

Convergent

(e)

$$\sum_{n=2}^{\infty}\frac{\ln(n)}{n}$$

Solution:

Divergent

3. (25 points) Determine the interval of convergence of the following power series:

$$\sum_{n=1}^{\infty} \frac{n!}{1 \cdot 3 \cdot 5 \cdots (2n-1)} (x+1)^n$$

Hint: $2 \cdot 4 \cdot 6 \cdots 2n = 2^n \cdot n!$.

Solution:

$$a_n = \frac{n!}{1 \cdot 3 \cdot 5 \cdots (2n-1)} (x+1)^n \Rightarrow \left| \frac{a_{n+1}}{a_n} \right| = \frac{n+1}{2n+1} |x+1|$$

$$\longrightarrow \frac{1}{2} |x+1| \quad \text{as } n \to \infty$$

If $\frac{1}{2}|x+1| < 1$ (ie $|x+1| < 2$) $\Rightarrow$ con

$\frac{1}{2}|x+1| > 1$ (ie $|x+1| > 2$) $\Rightarrow$ div

$\Rightarrow R = 2$ $\quad (-3, 1)$

$\underline{x = 1}$: $\displaystyle\sum_{n=1}^{\infty} \frac{2^n n!}{1 \cdot 3 \cdots (2n-1)}$

$$\frac{2^n n!}{1 \cdot 3 \cdots (2n-1)} = \frac{2 \cdot 4 \cdot 6 \cdots 2n}{1 \cdot 3 \cdot 5 \cdots 2n-1} > 1 \Rightarrow \text{Divergent}$$

Same logic for $x = -3$ $\Rightarrow$ I.O.C. $= (-3, 1)$

4. (25 points) Calculate the Maclaurin series of the following function.

$$f(x) = \frac{x}{(1-x)^3}$$

Be sure to include a general term.

Solution:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$$

$$\Rightarrow \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \ldots$$

$$\Rightarrow \frac{2}{(1-x)^3} = 2 + 3 \cdot 2 \cdot x + \ldots + (n+2)(n+1)\, x^n + \ldots$$

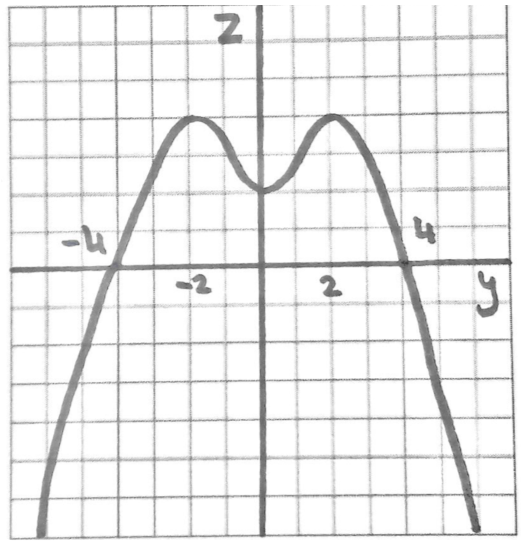
$$\Rightarrow \frac{x}{(1-x)^3} = \frac{1}{2} 2x + \frac{1}{2} \cdot 3 \cdot 2 \cdot x^2 + \ldots + \frac{1}{2}(n+1)\, n\, x^n + \ldots$$

for all x in $(-1, 1)$

Maclaurin Series for $\frac{x}{(1-x)^3}$

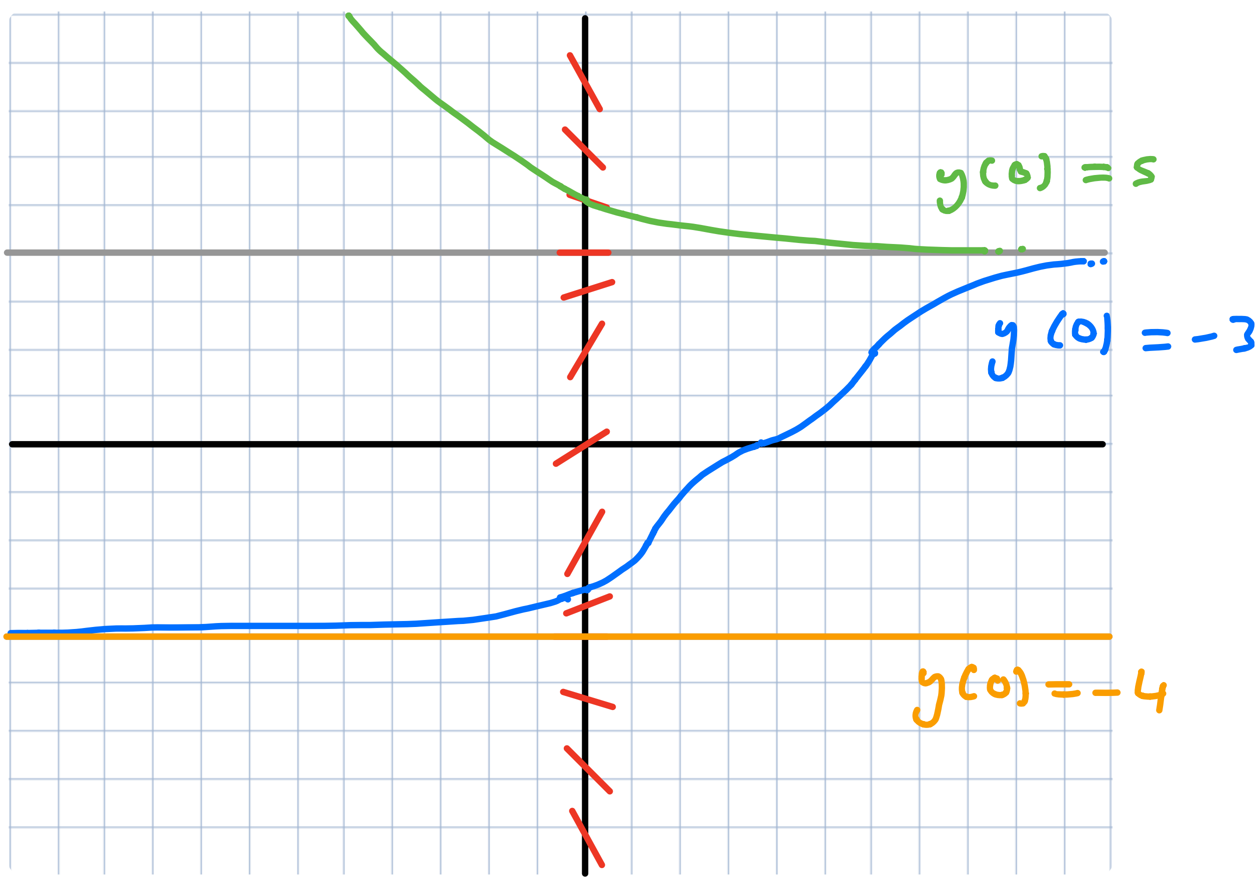
(You could also solve this problem using the binomial series)

5. (25 points) Consider the differential equation of the form $y' = h(y)$, where the graph $z = h(y)$ is as follows:



Sketch a solution for each of the following initial conditions: $y(0) = -3$, $y(0) = 5$ and $y(0) = -4$.

Solution:



6. (25 points) Find an equation for the orthogonal trajectory to the family of curves

$$y = \frac{k}{x-1} + 3 \qquad (k \text{ any constant})$$

which contains the point $(1, 1)$.

Solution:

$$\frac{dy}{dx} = \frac{-k}{(x-1)^2} \qquad k = (y-3)(x-1)$$

$$\Rightarrow \frac{dy}{dx} = \frac{-(y-3)}{x-1}$$

Must solve $\frac{dy}{dx} = \frac{x-1}{y-3} \Rightarrow \int (y-3)\,dy = \int (x-1)\,dx$

$$\Rightarrow \frac{1}{2}(y-3)^2 = \frac{1}{2}(x-1)^2 + C$$

$$\Rightarrow (y-3)^2 = (x-1)^2 + 2C$$

$$(1-3)^2 = (1-1)^2 + 2C \quad \Rightarrow 2C = 4$$

$$\Rightarrow (y-3)^2 = (x-1)^2 + 4$$

7. (25 points) Solve the following initial value problem:

$$xy' + y = x\ln(x), \qquad y(1) = 0, \quad x > 0$$

Solution:

$y' + \frac{1}{x} y = \ln(x)$. $\int \frac{1}{x} dx = \ln(x) + C \quad (x > 0)$

$\Rightarrow I(x) = x$

$\Rightarrow y = \frac{1}{x} \int x \ln(x)\, dx$

$f(x) = \ln(x)$, $g'(x) = x$

$f'(x) = \frac{1}{x}$, $g(x) = \frac{1}{2}x^2$

$\Rightarrow \int x \ln(x)\, dx = \frac{1}{2}x^2 \ln(x) - \int \frac{1}{2} x\, dx$

$= \frac{1}{2}x^2 \ln(x) - \frac{1}{4}x^2 + C$

$\Rightarrow y = \frac{1}{2} x \ln(x) - \frac{1}{4}x + \frac{C}{x}$

$y(1) = 0 \Rightarrow -\frac{1}{4} + C = 0 \Rightarrow C = \frac{1}{4}$

$\Rightarrow y = \frac{1}{2} x \ln(x) - \frac{1}{4}x + \frac{1}{4x}$.

8. (25 points) Solve the following initial value problem:

$$e^{\sin(x)} y' = \sin(\pi y), \qquad y(2) = 1$$

Solution:

$$e^{\sin(x)} \frac{dy}{dx} = \sin(\pi y) \Rightarrow \frac{dy}{dx} = \underbrace{e^{-\sin(x)}}_{g(x)} \cdot \underbrace{\sin(\pi y)}_{h(y)}$$

$h(y) = 0 \Leftrightarrow \sin(\pi y) = 0 \Leftrightarrow y = n$ an integer

$\Rightarrow y(x) = 1$ a constant solution

$\Rightarrow y(x) = 1$ for all x $\Rightarrow y(2) = 1$

$\Rightarrow y(x) = 1$ solves initial value problem.

9. (25 points) Find a general solution to the differential equation:

$$y'' + 2y' + y = e^{-x} + e^{x}$$

Solution:

Comp Eq: $y'' + 2y' + y = 0$, Char Eq: $(r+1)^2 = 0 \Rightarrow r = -1$

$\Rightarrow y_c = c_1 e^{-x} + c_2 x e^{-x}$

$y_{p_1} = A x^2 e^{-x}$

$y'_{p_1} = 2Axe^{-x} - Ax^2e^{-x}$

$y''_{p_1} = 2Ae^{-x} - 2Axe^{-x} - 2Axe^{-x} + Ax^2e^{-x}$

$\Rightarrow y''_{p_1} + 2y'_p + y_p = 2Ae^{-x}$

$\Rightarrow 2A = 1 \Rightarrow A = \frac{1}{2}$

$y_{p_2} = Ae^{x} \Rightarrow y''_{p_2} + 2y'_{p_2} + y_{p_2} = 4Ae^{x} \Rightarrow 4A = 1$

$\Rightarrow A = \frac{1}{4}$

$\Rightarrow y = c_1 e^{-x} + c_2 x e^{-x} + \frac{1}{2}x^2e^{-x} + \frac{1}{4}e^{x}.$

10. (25 points) Find a power series solution to the following initial value problem:

$$y'' - xy' = y, \qquad y(0) = 1, y'(0) = 0 \quad \Rightarrow \quad c_0 = 1, \; c_1 = 0$$

Solution:

$$y = c_0 + c_1 x + c_2 x^2 + \cdots = \sum_{n=0}^{\infty} c_n x^n$$

$$xy' = c_1 x + 2c_2 x^2 + \cdots = \sum_{n=1}^{\infty} n c_n x^n$$

$$y'' = \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n$$

$$\Rightarrow y'' - xy' - y = (2c_2 - c_0) + \sum_{n=1}^{\infty} \big((n+2)(n+1) c_{n+2} - (n+1) c_n\big) x^n$$

$$\Rightarrow 2c_2 - c_0 = 0 \Rightarrow c_2 = \frac{c_0}{2} \; ; \; c_{n+2} = \frac{c_n}{n+2} \quad n \geq 1$$

$\underline{n=1}$ $\quad c_3 = \frac{c_1}{3} = 0$ $\qquad$ $\underline{n=3}$ $\quad c_5 = \frac{c_3}{5} = 0$

$\underline{n=2}$ $\quad c_4 = \frac{c_2}{4} = \frac{c_0}{4 \cdot 2} = \frac{1}{4 \cdot 2}$ $\qquad$ $\underline{n=4}$ $\quad c_6 = \frac{c_0}{6 \cdot 4 \cdot 2} = \frac{c_0}{2^3 \cdot 3!} = \frac{1}{2^3 \cdot 3!}$

<u>Pattern</u>

$$c_{2k} = \frac{1}{2^k k!} \quad (0! = 1) \quad \text{for } k \geq 0$$

$$c_{2k+1} = 0 \quad k \geq 0$$

$$\Rightarrow y = \sum_{k=0}^{\infty} \frac{x^{2k}}{2^k k!}$$

END OF EXAM