Math 1B : Calculus [redacted]

[redacted] : Alternating Series.

An <u>alternating series</u> is a series whose terms alternate between positive and negative.

<u>Examples</u> :

$$\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{5} + \dots$$

$$\sum_{n=1}^{\infty} \frac{\sin\left(\frac{(2n+1)\pi}{2}\right)}{1+n^2} = -\frac{1}{2} + \frac{1}{5} - \frac{1}{10} + \dots$$

<u>Interesting question</u> : When does an alternating series converge?

<u>Alternating Series Test</u>

Let $\{a_n\}$ be a sequence of positive numbers such that

1) $a_{n+1} \leq a_n$ for all $n \geq 1$

2) $\lim_{n \to \infty} \{a_n\} = 0$

Then the alternating series $\sum_{n=1}^{\infty} (-1)^{n-1} a_n = a_1 - a_2 + a_3 - a_4 + \dots$ is convergent.

Remarks

1) <u>Warning</u>: This will only work if the series is alternating.

Recall that if $\{b_n\}$ is a sequence such that $\sum_{n=1}^{\infty} b_n$ is convergent, then this implies $\lim_{n\to\infty} \{b_n\} = 0$.

<u>However</u>, the converse is <u>not</u> true in general, meaning that just because $\lim_{n\to\infty} \{b_n\} = 0$, it does <u>not</u> follow that $\sum_{n=1}^{\infty} b_n$ is convergent. The standard example is the harmonic series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots$.

2) As with the integral test and the comparison tests, the alternating series test only tells us that a series converges [redacted]. It does <u>not</u> tell us what the value is if convergent.

Examples

1) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \dots = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{n}$

convergent by alternating series test, because

i) $\frac{1}{n+1} < \frac{1}{n}$ for all $n \geq 1$

ii) $\lim_{n\to\infty} \{\frac{1}{n}\} = 0$.

2) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n^2}{n^3+1}$

Observe that this is an alternating series. Let's try and use the alternating series test. Let $a_n = \dfrac{n^2}{n^3+1}$ for all $n \geq 1$. We first need to show $\{a_n\}$ is decreasing. Let's use calculus! The sequence obviously comes from the function $f(x) = \dfrac{x^2}{x^3+1}$.

$f'(x) = \dfrac{x(2-x^3)}{(x^3+1)^2}$. Note that for $x > \sqrt[3]{2}$, $f'(x) < 0$, hence $f(x)$ is decreasing on $(\sqrt[3]{2}, \infty)$.

Thus $a_2 > a_3 > a_4 > a_5 \cdots$.

We could also verify that $a_1 > a_2$, but all that really matters is that $\{a_n\}$ is <u>eventually</u> decreasing.

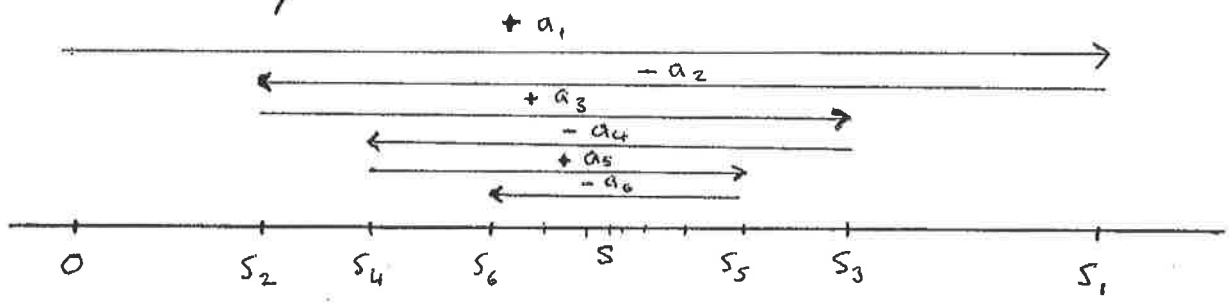
Now observe that

$$\lim_{n\to\infty} \{a_n\} = \underbrace{\lim_{n\to\infty} \left\{\frac{n^2}{n^3+1}\right\}}_{\text{limit of sequence}} = \underbrace{\lim_{x\to\infty} \frac{x^2}{x^3+1}}_{\text{limit of function}} \underset{\text{L'Hopitales done twice.}}{=} 0$$

Hence <u>convergent</u> by alternating series test.

Why is the alternating series test true?

Let's draw a picture:



$$S_1 = a_1$$
$$S_2 = a_1 - a_2$$
$$S_3 = a_1 - a_2 + a_3$$
$$S_4 = a_1 - a_2 + a_3 - a_4$$
$$\vdots$$

Consider the sequence of even partial sums:

$$\{S_2, S_4, S_6, S_8, \dots\}$$

Note that this sequence is <u>increasing</u> and bounded above by $S_1 = a_1$. Hence by the monotone sequence theorem (§11.1) it is convergent with limit which we denote by S.

Recall that $\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ is the limit of the whole sequence $\{S_1, S_2, S_3, S_4, \dots\}$. We need to think about the odd terms also.

Notice that

$$S_{2n+1} = S_{2n} + a_{2n+1} \quad , \text{ hence}$$

$$\begin{aligned}
\lim_{n\to\infty} \{S_{2n+1}\} &= \lim_{n\to\infty} \{S_{2n} + a_{2n+1}\} \\
&= \lim_{n\to\infty} \{S_{2n}\} + \lim_{n\to\infty} \{a_{2n+1}\} \\
&= S + 0 \\
&= S.
\end{aligned}$$

Since both even and odd partial sums converge to S, we deduce that $\lim_{n\to\infty} \{S_n\} = S$ and that

$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$ converges.

A useful thing we can see from this is that

if $S = \sum_{n=1}^{\infty} (-1)^{n-1} a_n$, where

i) $a_{n+1} \leq a_n$ $\quad (a_n \geq 0)$

ii) $\lim_{n\to\infty} \{a_n\} = 0$

E.g. n even

then $|S - S_n| \leq a_{n+1}$

We can use this in the following way:

Estimate the sum $\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{n^6}$ with an error of less that $\frac{1}{1000}$.

First note that the series is convergent by the alternating series test.

$S_1 = 1$, $\quad |S - S_1| \leq a_2 = 0.015625$

$S_2 = 1 - \frac{1}{2^6}$, $\quad |S - S_2| \leq a_3 = 0.0013717$

$S_3 = 1 - \frac{1}{2^6} + \frac{1}{3^6}$, $\quad |S - S_3| \leq a_4 = 0.000244 < \frac{1}{1000}$

Hence $S_3 = 1 - \frac{1}{2^6} + \frac{1}{3^6} \approx 0.9857$.

Hence $S \approx 0.9857$ to within $\frac{1}{1000}$.

<u>Warning</u> This only works because the series is alternating. It would not work for general sequences.