1A Homework 9 Solutions

§4.3

Q1/ a) $(1,3), (4,6)$ b) $(0,1), (3,4)$ c) $(0,2)$ d) $(2,4), (4,6)$ e) $(2,3)$

Q6/ a) Increasing : $[2,3], [4,6]$

Decreasing : $[0,2], [3,4], [6,8]$

b) $x = 2, 4$ give local min.

$x = 3, 6$ give local max

Q8/ a) f increasing on $(0,4)$ and $(6,8)$ as $f'(x) > 0$.

b) $x = 6$ gives local min as $f'(x)$ switches from $-$ to $+$ and $f'(x) = 0$

$x = 4, 8$ give local max as $f'(x)$ switches from $+$ to $-$ and $f'(x) = 0$

c) $f'(x)$ increasing on $(0,1), (2,3)$ and $(5,7)$ $\Rightarrow$ concave up

$f'(x)$ decreasing on $(1,2), (3,5)$ and $(7,9)$ $\Rightarrow$ concave down

d) $x = 1, 2, 3, 5, 7$ are inflection points as it switches concavity at all these points.

Q12/ $f(x) = \dfrac{x}{x^2+1} \Rightarrow f'(x) = \dfrac{x^2+1 - 2x^2}{(x^2+1)^2} = \dfrac{1-x^2}{(1+x^2)^2}$

$f'(x) = 0 \Leftrightarrow x = \pm 1$

$$\begin{array}{ccccc} - & & + & & - \\ \hline & -1 & & 1 & \end{array} \Rightarrow$$

Decreasing on $(-\infty, -1)$ and $(1, \infty)$. Increasing on $(-1, 1)$

$f(-1) = \frac{-1}{2}$ local min

$f(1) = \frac{1}{2}$ local max

$$f''(x) = \frac{-2x(1+x^2)^2 - 4x(1+x^2)(1-x^2)}{(1+x^2)^4}$$

$$= \frac{-2x(1+x^2) - 4x(1-x^2)}{(1+x^2)^3} = \frac{2x(x^2-3)}{(1+x^2)^3}$$

$f''(x) = 0 \Leftrightarrow x = 0, \sqrt{3}, -\sqrt{3}$

$$\begin{array}{ccccccc} - & & + & & - & & + \\ \hline & -\sqrt{3} & & 0 & & \sqrt{3} & \end{array} \Rightarrow$$

Concave up on $(-\sqrt{3}, 0)$, $(\sqrt{3}, \infty)$

Concave down on $(-\infty, -\sqrt{3})$, $(0, \sqrt{3})$

$0, \pm\sqrt{3}$ inflection points.

Q21/ $f(x) = \sqrt{x} - \sqrt[4]{x} = x^{\frac{1}{2}} - x^{\frac{1}{4}}$ $(x \geqslant 0)$

$\Rightarrow f'(x) = \frac{1}{2} x^{-\frac{1}{2}} - \frac{1}{4} x^{-\frac{3}{4}}$ $(x > 0)$

$f'(x) = 0 \Rightarrow \frac{1}{2} x^{-\frac{1}{2}} - \frac{1}{4} x^{-\frac{3}{4}} = 0$

$\Rightarrow \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{4} x^{-\frac{3}{4}}$

$\Rightarrow \frac{1}{2} x^{3/4} = \frac{1}{4} x^{\frac{1}{2}}$

$\Rightarrow \left(\frac{1}{2}\right)^4 x^3 = \left(\frac{1}{4}\right)^4 \cdot x^2$

$\Rightarrow \left(\frac{1}{16}\right) x^3 = \frac{1}{256} x^2$

$\Rightarrow x = \frac{16}{256} = \frac{1}{16}$

$f\left(\frac{1}{81}\right) < 0$, $f(1) > 0$

$f'(x)$: $-$ on $(0, 1/16)$, $+$ after $1/16$ ← local min

$f''\left(\frac{1}{16}\right) = \frac{1}{2} \cdot \frac{-1}{2} \left(\frac{1}{16}\right)^{-\frac{3}{2}} + \frac{3}{16} \left(\frac{1}{16}\right)^{-\frac{7}{4}}$

$= \frac{-1}{4} \cdot (16)^{\frac{3}{2}} + \frac{3}{16} (16)^{\frac{7}{4}}$

$= \frac{-1}{4} 4^3 + \frac{3}{16} 2^7$

$= -16 + 3 \cdot 8 = 8 > 0 \Rightarrow$ local min

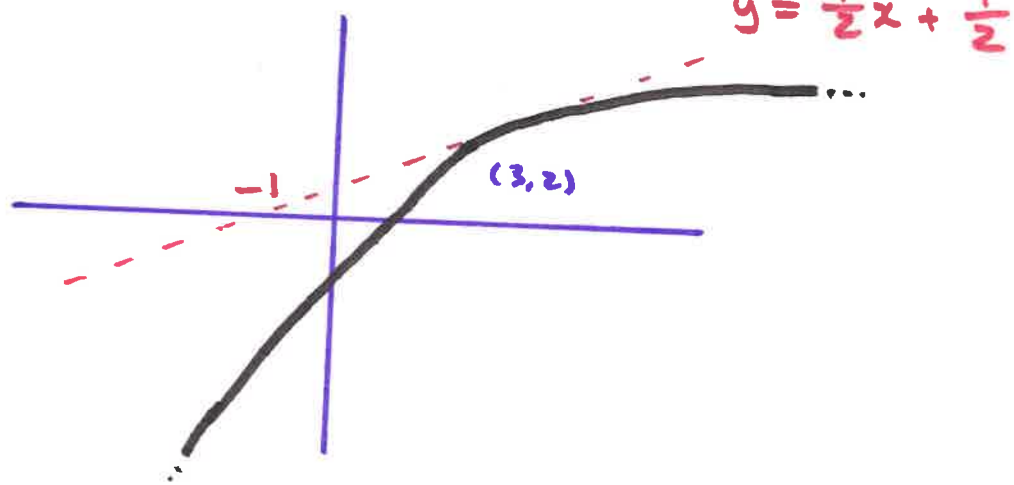
Q23

a) $f(2)$ is a local max

b) Nothing

Q32

a)



b) Looking at graph it seems there will be one solution.

Formal Proof (Hard):

1/ Lets show there is at least one root using I.V.T.

$f(3) = 2 > 0$. Need to find a such that $f(a) < 0$ then apply I.V.T.

Claim : $f(-2) < 0$

Proof : Assume not, ie $f(-2) \geqslant 0$. Consider

$$\frac{f(3) - f(-2)}{3-(-2)} = \frac{2 - f(-2)}{5}.$$ Because $f(-2) \geqslant 0$

This quantity is less than or equal $\frac{2}{5}$

By M.V.T. There is a point c in $(-2, 3)$ such that

$$f'(c) = \frac{f(3) - f(-2)}{3-(-2)} \leq \frac{2}{5}$$

Now consider $$\frac{f'(3) - f'(c)}{3-c} = \frac{\frac{1}{2} - f'(c)}{3-c}$$

$$f'(c) \leq \frac{2}{5} \Rightarrow \frac{1}{2} - f'(c) > 0 \Rightarrow \frac{f'(3) - f'(c)}{3-c} > 0$$

But by M.V.T. there is a d in $(c, 3)$ with

$$f''(d) = \frac{f'(3) - f'(c)}{3-c} > 0.$$

This is not possible as $f''(x) < 0$ for all x in $\mathbb{R}$.

Hence $f(-2) < 0$.

Thus by I.V.T. there is a root between -2 and 3.

2/ Let's show there is at most one root using M.V.T.

Assume $a < b$ and $f(a) = f(b) = 0$

$\Rightarrow$ there exists c in (a, b) with $f'(c) = 0$
(Rolle's Theorem)

This is impossible as $f'(x) > 0$ for all x in $\mathbb{R}$

Thus there <u>cannot</u> be 2 distinct roots.

c) $f'(2) = \frac{1}{3} \Rightarrow \frac{f'(3) - f'(2)}{3-2} = \frac{\frac{1}{2} - \frac{1}{3}}{3-1} = \frac{1}{6} > 0$

By M.V.T. there exists c in $(2,3)$ with $f'(c) = \frac{1}{6} > 0$. This is not possible as $f''(x) < 0$ for all x. Therefore $f'(2) \neq \frac{1}{3}$.

Q34 a) B, b) E, c) A

Q47 $f(\theta) = 2\cos\theta + \cos^2\theta \Rightarrow f'(\theta) = -2\sin\theta - 2\sin\theta\cos\theta$

$0 \leq \theta \leq 2\pi$

$f'(\theta) = 0 \Leftrightarrow \sin\theta = 0$ or $\cos\theta = -1$

$\Rightarrow \theta = \pi$

$-$ $+$ $f'(\theta)$ $\Rightarrow f(\pi)$ local min

$0 \quad \pi \quad 2\pi$

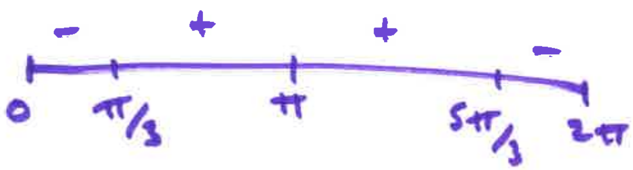
$f''(\theta) = -2\cos\theta - 2\cos^2\theta + 2\sin^2\theta = -2\cos\theta - 2\cos^2\theta + 2(1-\cos^2\theta)$

$= -4\cos^2\theta - 2\cos\theta + 2 = -2(2\cos^2\theta + \cos\theta - 1)$

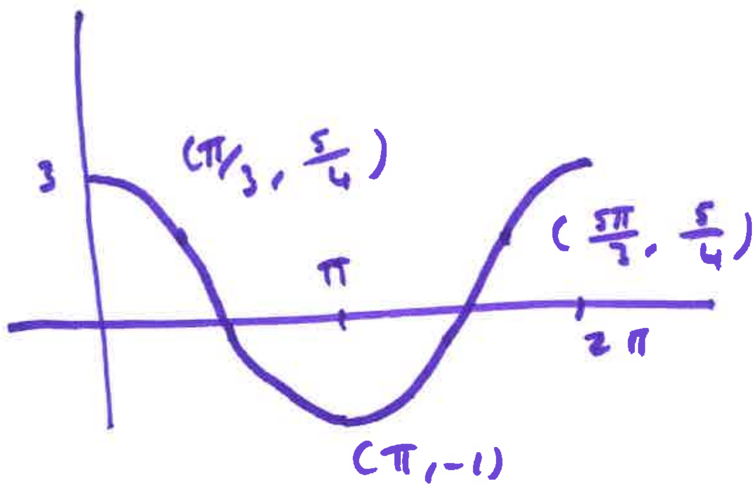
$$= -2(2\cos\theta - 1)(\cos\theta + 1)$$

$$f''(\theta) = 0 \implies \cos\theta = \frac{1}{2} \text{ or } \cos\theta = -1$$

$$\implies \theta = \frac{\pi}{3},\ \frac{5\pi}{3} \text{ or } \pi$$



$f''(\theta) \implies \pi/3$ and $5\pi/3$ inflections



Q66 $f'(x) < 0$, $f''(x) > 0$

Q77 $f(x) = \dfrac{1+x}{1+x^2} \implies f'(x) = \dfrac{(1+x^2) - 2x(1+x)}{(1+x^2)^2}$

$$\Rightarrow \quad f'(x) = \frac{1-2x-x^2}{(1+x^2)^2}$$

$$\Rightarrow \quad f''(x) = \frac{(-2-2x)(1+x^2)^2 - 4x(1+x^2)(1-2x-x^2)}{(1+x^2)^4}$$

$$= \frac{(-2-2x)(1+x^2) - 4x(1-2x-x^2)}{(1+x^2)^3}$$

$$= \frac{-2-2x^3-2x-2x^2-4x+8x^2+4x^3}{(1+x^2)^3}$$

$$= \frac{2x^3+6x^2-6x-2}{(1+x^2)^3} = \frac{(x-1)(2x^2+8x+2)}{(1+x^2)^3}$$

$$f''(x) = 0 \iff x = 1 \text{ or } x = -2 \pm \sqrt{3}$$

$$\begin{array}{ccccccc} - & & + & & - & & + \\ \hline & -2-\sqrt{3} & & -2+\sqrt{3} & & 1 & \end{array} \quad f''(x) \quad \Rightarrow -2-\sqrt{3},\ -2+\sqrt{3} \text{ and } 1 \text{ inflection points.}$$

$$\frac{f(1) - f(-2+\sqrt{3})}{1-(-2+\sqrt{3})} = \frac{1 - \dfrac{1+(-2+\sqrt{3})}{1+(-2+\sqrt{3})^2}}{1-(-2+\sqrt{3})} \overset{\text{Long calculation}}{=} \frac{1}{4}$$

$$\frac{f(-2+\sqrt{3}) - f(-2-\sqrt{3})}{(-2+\sqrt{3})-(-2-\sqrt{3})} \overset{\text{Long calculation}}{=} \frac{1}{4}$$

$\Rightarrow$ $(1, f(1))$, $(-2+\sqrt{3}, f(-2+\sqrt{3}))$ and $(-2-\sqrt{3}, f(-2-\sqrt{3}))$

are colinear.

Q84

$$f(x) = x|x| = \begin{cases} x^2 & x \geq 0 \\ -x^2 & x < 0 \end{cases}$$

$$f'(0) = \lim_{h\to 0} \frac{h|h|}{h} = 0 \Rightarrow f'(x) = \begin{cases} 2x & x > 0 \\ 0 & x = 0 \\ -2x & x < 0 \end{cases} = 2|x|$$

$$f''(0) = \lim_{h\to 0} \frac{2|h|}{h} \quad \text{DNE} \Rightarrow f''(x) = \begin{cases} 2 & x > 0 \\ \text{DNE} & x = 0 \\ -2 & x < 0 \end{cases}$$

$\Rightarrow$ $f(x)$ concave up for $x>0$ <u>and</u> concave down for $x<0$ <u>and</u>

$f(x)$ is continuous at 0 $\Rightarrow$ $x=0$ is inflection.

However $f''(0)$ DNE.

§4.4.

Q8/ $$\lim_{x \to 3} \frac{x-3}{x^2-9} = \lim_{x \to 3} \frac{1}{2x} = \frac{1}{6}$$

Q16/ $$\lim_{x \to 0} \frac{x^2}{1-\cos(x)} = \lim_{x \to 0} \frac{2x}{\sin(x)} = \lim_{x \to 0} \frac{2}{\cos(x)} = 2$$

Q19/ $$\lim_{x \to \infty} \frac{\ln(x)}{\sqrt{x}} = \lim_{x \to \infty} \frac{1/x}{1/2\sqrt{x}} = \lim_{x \to \infty} \frac{2}{\sqrt{x}} = 0$$

Q27/ $$\lim_{x \to 0} \frac{e^x - 1 - x}{x^2} = \lim_{x \to 0} \frac{e^x - 1}{2x} = \lim_{x \to 0} \frac{e^x}{2} = \frac{1}{2}$$

Q46/ $$\lim_{x \to -\infty} x \ln\left(1 - \frac{1}{x}\right) = \lim_{x \to -\infty} \frac{\ln(1-\frac{1}{x})}{1/x} = \lim_{x \to -\infty} \left(\frac{\frac{\frac{+1}{x^2}}{1 - 1/x}}{-1/x^2} \right)$$

$$= \lim_{x \to -\infty} \frac{-1}{1 - 1/x} = -1$$

Q54/ $$\lim_{x \to 0^+} \left(\frac{1}{x} - \frac{1}{\arctan(x)} \right) = \lim_{x \to 0^+} \frac{\arctan(x) - x}{x\arctan(x)}$$

$$= \lim_{x \to 0^+} \frac{\frac{1}{1+x^2} - 1}{\arctan(x) + \frac{x}{1+x^2}} = \lim_{x \to 0^+} \frac{-x^2}{(1+x^2)\arctan(x) + x}$$

$$= \lim_{x \to 0^+} \frac{-2x}{2x\arctan(x) + 2} = 0$$

Q57 $$\lim_{x \to 0^+} \ln\left(x^{\sqrt{x}}\right) = \lim_{x \to 0^+} \sqrt{x}\,\ln(x) = \lim_{x \to 0^+} \frac{\ln(x)}{1/\sqrt{x}}$$

$$= \lim_{x \to 0^+} \frac{1/x}{-\frac{1}{2}x^{-3/2}} = \lim_{x \to 0^+} -2\sqrt{x} = 0 \Rightarrow \lim_{x \to 0^+} x^{\sqrt{x}} = e^0 = 1$$

Q60 $$\lim_{x \to \infty} \ln\left(\left(1+\frac{a}{x}\right)^{bx}\right) = \lim_{x \to \infty} bx\,\ln\left(1+\frac{a}{x}\right)$$

$$= \lim_{x \to \infty} \frac{\ln\left(1+\frac{a}{x}\right)}{1/bx} = \lim_{x \to \infty} \frac{\left(\dfrac{\frac{-a}{x^2}}{1+a/x}\right)}{\left(\frac{-1}{bx^2}\right)} = \lim_{x \to \infty} \frac{ab}{1+\frac{a}{x}}$$

$$= ab \Rightarrow \lim_{x \to \infty} \left(1+\frac{a}{x}\right)^{bx} = e^{ab}$$

Q64 $$\lim_{x \to \infty} \ln\left(x^{e^{-x}}\right) = \lim_{x \to \infty} e^{-x}\ln(x) = \lim_{x \to \infty} \frac{\ln(x)}{e^x}$$

$$= \lim_{x \to \infty} \frac{1}{xe^x} = 0 \Rightarrow \lim_{x \to \infty} x^{e^{-x}} = e^0 = 1$$

Q86/

$$\lim_{x \to a} f(x) = 0^+ \implies \lim_{x \to a} \ln(f(x)) = -\infty$$

$$\lim_{x \to a} g(x) = \infty \implies \lim_{x \to a} g(x) \ln(f(x)) = -\infty$$

$$\implies \lim_{x \to a} \ln\left(f(x)^{g(x)}\right) = -\infty$$

$$\implies \lim_{x \to a} f(x)^{g(x)} = \lim_{x \to a} e^{\ln\left(f(x)^{g(x)}\right)} = 0$$

Q88/

$$\frac{\sin(2x)}{x^3} + \frac{b}{x^2} = \frac{\sin(2x) + bx}{x^3}$$

$$\lim_{x \to 0} \frac{\sin(2x) + bx}{x^3} = \lim_{x \to 0} \frac{2\cos(2x) + b}{3x^2}$$

$$\lim_{x \to 0} 2\cos(2x) + b = 2 + b$$

If $2 + b \neq 0 \implies \lim_{x \to 0} \frac{2\cos(2x) + b}{3x^2} = \pm\infty$

$\implies$ Need $b = -2$ at the least

Assume $b = -2$.

$$\Rightarrow \lim_{x \to 0} \frac{2\cos(2x) - 2}{3x^2} = \lim_{x \to 0} \frac{-4\sin(2x)}{6x} = \lim_{x \to 0} \frac{-4}{3} \cdot \frac{\sin(2x)}{2x}$$

$$= \frac{-4}{3}.$$

Hence need $b = -2$ and $a = \frac{4}{3}$.

§ 4.5

Q4 $f(x) = x^4 - 8x^2 + 8$

Domain $= \mathbb{R}$

Even Function

$\lim_{x \to \infty, -\infty} f(x) = \infty$. No vertical asymptotes

$f'(x) = 4x^3 - 16x = 4x(x^2 - 4) = 4x(x-2)(x+2)$

$-$ $+$ $-$ $+$

-2 0 2

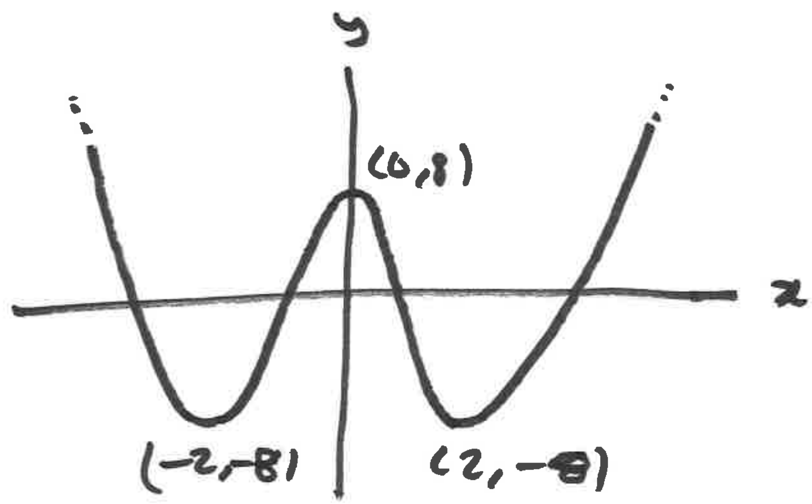
$-2, 2$ local min

0 local max

$f''(x) = 12x^2 - 16 \Rightarrow f''(x) = 0$ when $x = \pm\frac{4}{\sqrt{12}} = \pm\frac{2}{\sqrt{3}}$

$+$ $-$ $+$

$-\frac{2}{\sqrt{3}}$ $\frac{2}{\sqrt{3}}$



Q10/ $f(x) = \dfrac{x^2+5x}{25-x^2} = \dfrac{x(x+5)}{(x+5)(5-x)} = \begin{cases} \dfrac{x}{5-x} & x \neq -5 \\ \text{undefined at } x = -5 \end{cases}$

Sketch $\dfrac{x}{5-x}$ and then remove part at $x=-5$

Domain : $\mathbb{R}$ minus 5

x, y - intercept $= (0,0)$

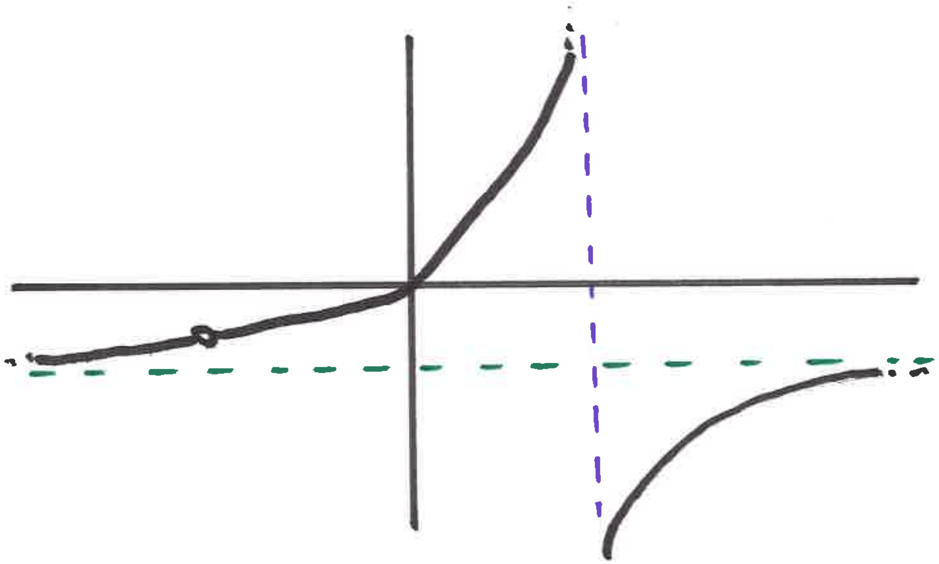
$\lim\limits_{x \to \pm\infty} \dfrac{x}{5-x} = -1 \Rightarrow y=-1$ horizontal asymptote

$\lim\limits_{x \to 5^+} \dfrac{x}{5-x} = -\infty$, $\lim\limits_{x \to 5^-} \dfrac{x}{5-x} = \infty \Rightarrow x=5$ vertical asymptote

$\dfrac{d}{dx}\left(\dfrac{x}{5-x}\right) = \dfrac{(5-x)-(-1)x}{(5-x)^2} = \dfrac{5}{(5-x)^2} \neq 0$

$+ \quad \underset{5}{|} \quad + \quad \Rightarrow$ No local max / min

$\dfrac{d^2}{dx^2}\left(\dfrac{x}{5-x}\right) = \dfrac{+10}{(5-x)^3}$ $\quad + \quad \underset{5}{|} \quad -$ ← not inflection.



Q24/ $f(x) = x - 3x^{1/3}$

Domain $= \mathbb{R}$

y - intercept $= (0,0)$

$x - 3x^{1/3} = 0 \Rightarrow x^3 = 3^3 x \Rightarrow x = 0, \pm\sqrt{27}$

Odd Function

$f(x) = x^{1/3}(x^{2/3} - 3) \Rightarrow \lim_{x \to \infty} f(x) = \infty$

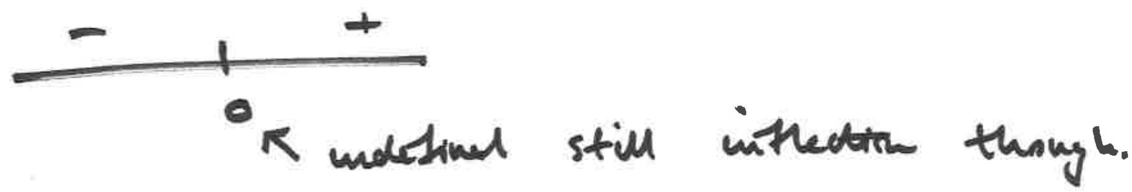
No vertical asymptote $\quad \lim_{x \to -\infty} f(x) = -\infty$

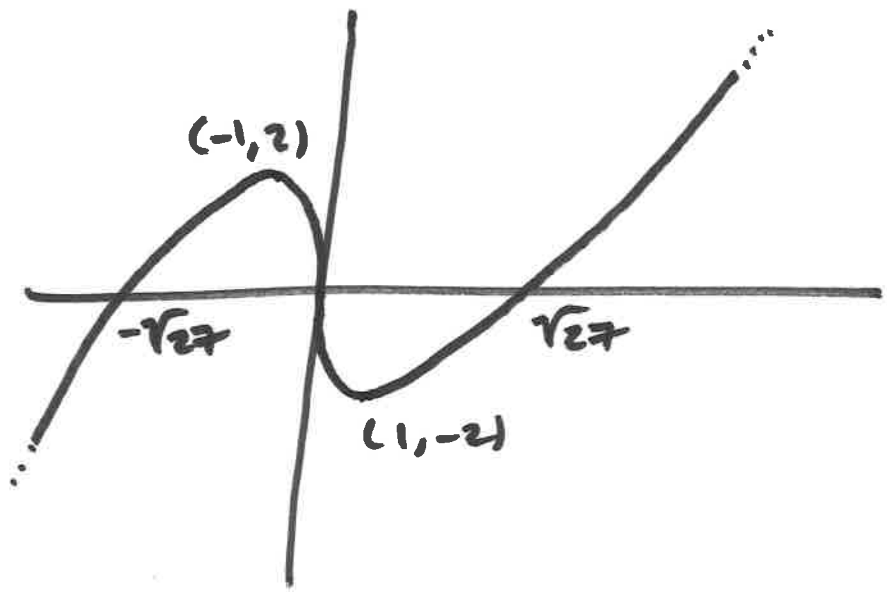
$f'(x) = 1 - x^{-2/3} = 0 \Rightarrow 1 = x^{-2/3} \Rightarrow 1 = \frac{1}{x^2}$

$\Rightarrow x = \pm 1$

$+$	-1	$-$	0	$-$	1	$+$

$$f''(x) = \frac{2}{3} x^{-5/3}$$





Q34

$$f(x) = \frac{\sin(x)}{1+\cos(x)}$$

Domain $= \mathbb{R}$ minus $\pi + 2k\pi$ (k integer), x, y-intercept $= (0,0)$

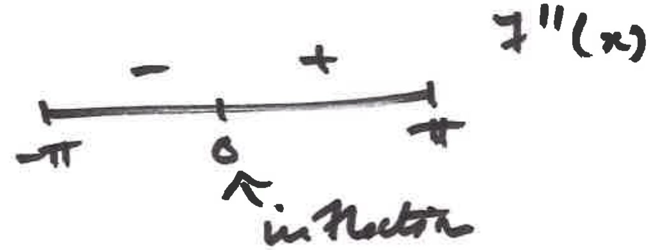
Periodic with period 2π. Lets graph it on $[-\pi, \pi]$.

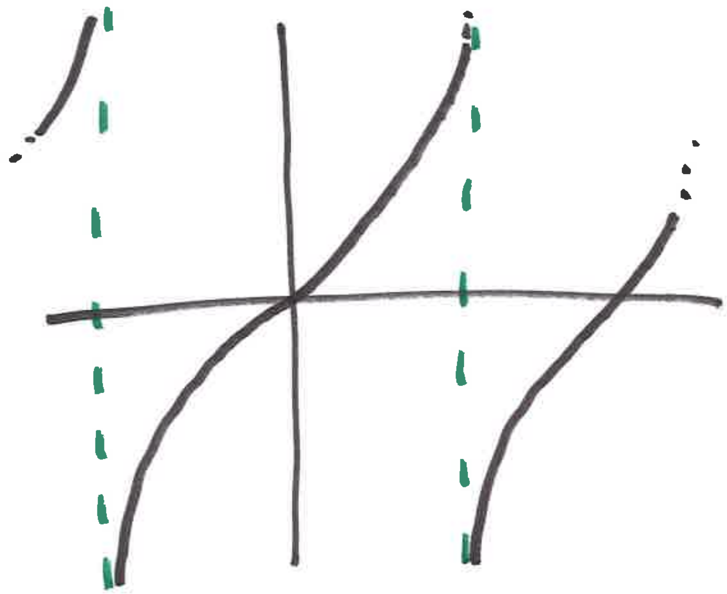
$\lim_{x \to -\pi^+} f(x) = -\infty$, $\lim_{x \to \pi^-} f(x) = \infty$

$$f'(x) = \frac{\cos(x)\,(1+\cos(x)) + \sin^2(x)}{(1+\cos(x))^2} = \frac{\cos(x)+1}{(\cos(x)+1)^2}$$

$$= \frac{1}{(\cos(x)+1)} > 0 \quad \text{on} \quad (-\pi, \pi)$$

$$f''(x) = \frac{-\sin(x)}{(\cos(x)+1)^2}$$





Q53 $f(x) = e^{\arctan(x)}$

Domain $= \mathbb{R}$ y-intercept $= (0,1)$. No x-intercept

$$\lim_{x\to\infty} e^{\arctan(x)} = e^{\pi/2}$$

$$\lim_{x\to-\infty} e^{\arctan(x)} = e^{-\pi/2}$$

$\Rightarrow$ $y = \frac{\pi}{2}$, $y = \frac{-\pi}{2}$ horizontal asymptotes

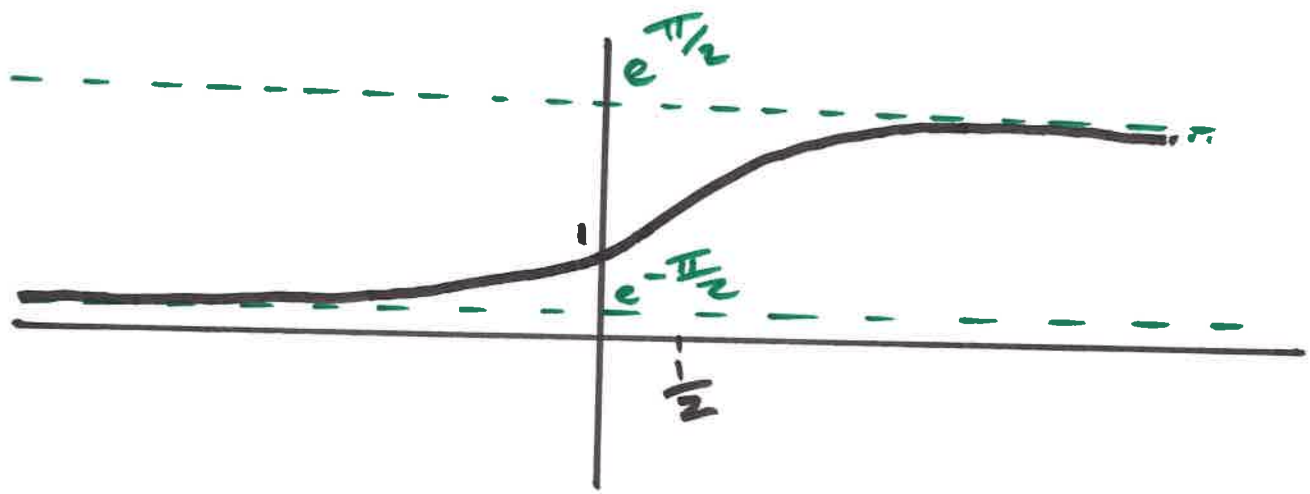
$$f'(x) = \frac{1}{1+x^2} e^{\arctan(x)} > 0$$

$$f''(x) = \frac{e^{\arctan(x)}}{(1+x^2)^2} + \frac{-2x}{(1+x^2)^2} e^{\arctan(x)}$$

$$= (1-2x) \cdot \frac{e^{\arctan(x)}}{(1+x^2)^2}$$

$+$ $\quad$ $-$

$\frac{1}{2}$ ← inflection



Q63

$$m = \frac{2}{1} = 2$$

$$\lim_{x \to \substack{+\infty \\ -\infty}} \left(\frac{2x^3 - 5x^2 + 3x}{x^2 - x - 2} - 2x \right) = \lim_{x \to \substack{+\infty \\ -\infty}} \frac{2x^3 - 5x^2 + 3x - 2x^3 + 2x^2 + 4x}{x^2 - x - 2}$$

$$= \lim_{x \to \substack{\infty \\ -\infty}} \frac{-3x^2 + 7x}{x^2 - x - 2} = -3$$

$\Rightarrow$ $y = 2x - 3$ is slant asymptote.

Q67

$$y = \frac{x^3 + 4}{x^2} = x + \frac{4}{x^2}, \quad \text{Domain} : \mathbb{R} \setminus \{0\}$$

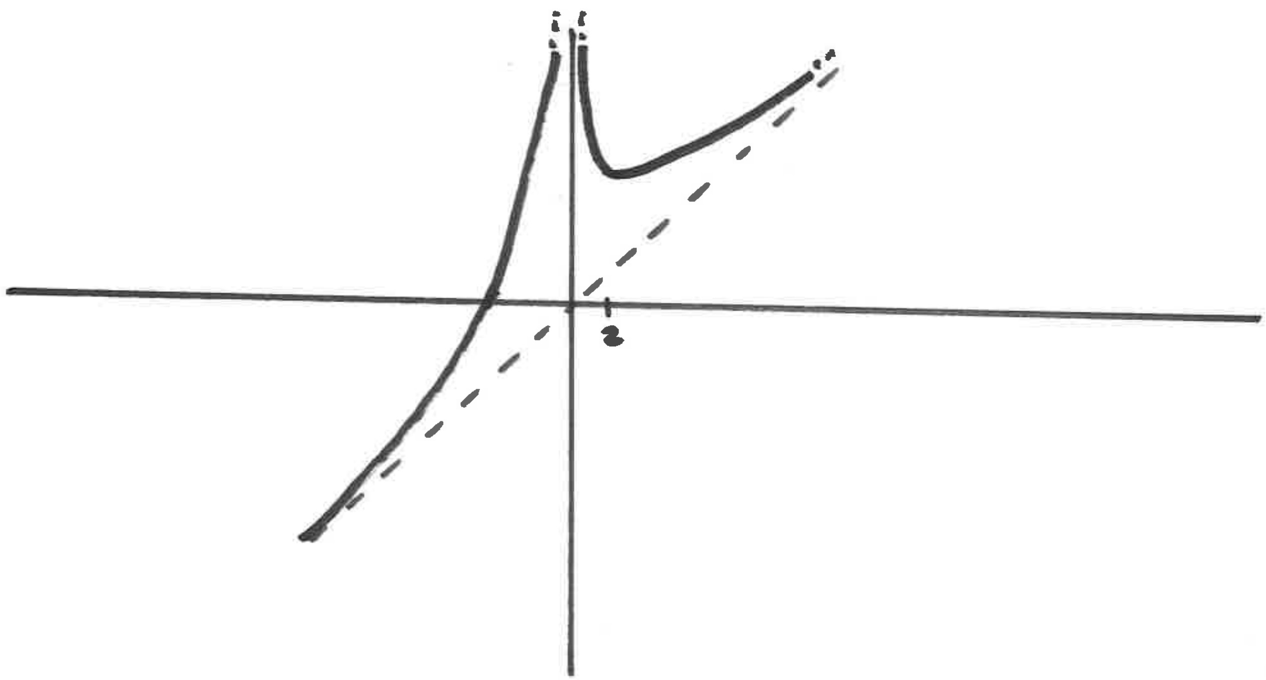
Slant asymptote : $y = x$

$$\lim_{x \to 0^+} x + \frac{4}{x^2} = \infty = \lim_{x \to 0^-} x + \frac{4}{x^2}$$

$$\frac{dy}{dx} = 1 - \frac{8}{x^3} \Rightarrow \frac{dy}{dx} = 0 \Leftrightarrow x = 2$$

$$\underset{}{-} \;\underset{2}{|}\; \underset{}{+}$$

$$\frac{d^2y}{dx^2} = \frac{24}{x^4} > 0$$



Q71

$$\left.\begin{aligned} &\lim_{x\to\infty}\left(x-\arctan(x)-\left(x-\frac{\pi}{2}\right)\right) \\ &= \lim_{x\to\infty}\left(-\arctan(x)+\frac{\pi}{2}\right) = 0 \\ &\lim_{x\to-\infty}\left(x-\arctan(x)-\left(x+\frac{\pi}{2}\right)\right) \\ &= \lim_{x\to-\infty}\left(-\frac{\pi}{2}-\arctan(x)\right) = 0 \end{aligned}\right\} \begin{aligned} &x+\frac{\pi}{2} \\ &\text{and} \\ &x-\frac{\pi}{2} \\ &\text{slant asymptotes.} \end{aligned}$$