1A homework 8 solutions

§3.8

Q1 $P'(t) = 0.7944\,P(t)$ and $P(0) = 2$

$\Rightarrow P(t) = 2e^{0.7944t} \Rightarrow P(6) = 2 \cdot e^{(0.7944)\cdot 6} \approx 235$

Q3 a) $P(t) = 100e^{kt}$ and $P(1) = 420$

$\Rightarrow 100e^{k} = 420 \Rightarrow k = \ln\left(\frac{420}{100}\right)$

$\Rightarrow P(t) = 100\,e^{\ln\left(\frac{420}{100}\right)t}$

b) $P(3) = 100\,e^{\ln\left(\frac{420}{100}\right)\cdot 3}$

c) $P'(3) = 100 \cdot \ln\left(\frac{420}{100}\right) e^{\ln\left(\frac{420}{100}\right)\cdot 3}$

d) $P(t) = 10000 \Rightarrow 100\,e^{\ln\left(\frac{420}{100}\right)t} = 10000 \Rightarrow t = \frac{\ln(10)}{\ln\left(\frac{420}{100}\right)}$

Q7 a) $\frac{d[N_2O_5]}{dt} = -0.0005\,[N_2O_5]$

$\Rightarrow [N_2O_5](t) = C\,e^{-0.0005t}$

b) $C\,e^{-0.0005t} = (0.9)\cdot C \Rightarrow t = \frac{\ln(0.9)}{-0.0005}$

Q11/

$$\left.\begin{aligned} m(t) &= m_0 e^{kt} \\ k &= \frac{-\ln(2)}{5730} \end{aligned}\right\} \Rightarrow m(t) = m_0 e^{-\frac{\ln(2)}{5730} t}$$

$$m(t) = 0.74 m_0 \Rightarrow 0.74 m_0 = m_0 e^{-\frac{\ln(2)}{5730} t}$$

$$\Rightarrow t = \frac{\ln(0.74)}{\left(\frac{-\ln(2)}{5730}\right)} \text{ years old}$$

Q12/

$$m(t) = m_0 e^{-\frac{\ln(2)}{5730} t}$$

$$m(68{,}000{,}000) = m_0 \cdot e^{-\frac{\ln(2)}{5730} \cdot 68000000}$$

There would be $e^{-\frac{\ln(2)}{5730} \cdot 68000000}$ ← very very small of the original ^{14}C.

$$m(t) = 0.01 m_0 \Rightarrow 0.01 = e^{-\frac{\ln(2)}{5730} t}$$

$$\Rightarrow t = \frac{\ln(0.01)}{\left(\frac{-\ln(2)}{5730}\right)} \approx 57{,}000 \text{ years}$$

↑ max age where we could use Carbon Dating.

Q16 $T(t) = T_s + (T_0 - T_s)e^{kt}$

$T_s = 20°C$ $\quad$ T_0: temperature of body at $t=0$

1.30 pm $\rightsquigarrow$ $t=0 \Rightarrow T_0 = 32.5°C$

2.30 pm $\rightsquigarrow$ $t=1 \Rightarrow T(1) = 30.3°C$

$\Rightarrow T(t) = 20 + (32.5 - 20)e^{kt}$

$\Rightarrow T(t) = 20 + 12.5e^{kt}$

$T(1) = 30.3 \Rightarrow 30.3 = 20 + 12.5e^{k}$

$\Rightarrow k = \ln\left(\frac{10.3}{12.5}\right)$

$\Rightarrow T(t) = 20 + 12.5e^{\ln\left(\frac{10.3}{12.5}\right)t}$

$T(t) = 37°C \Rightarrow 37 = 20 + 12.5e^{\ln\left(\frac{10.3}{12.5}\right)t}$

$\Rightarrow t = \dfrac{\ln\left(\frac{17}{12.5}\right)}{\ln\left(\frac{10.3}{12.5}\right)} \approx -0.794$ hours ≈ -48 minutes

$\Rightarrow$ Time of death ≈ 12.42 pm.

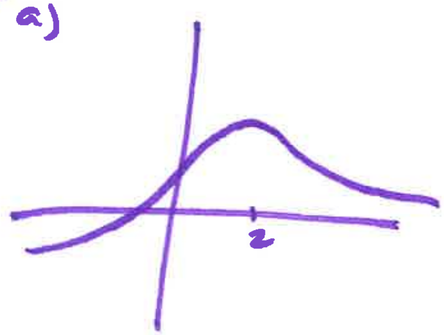
§4.1

Q3 s = Absolute Max
r = Absolute Min
c = Local Max
b, r = Local Min
a, d = Neither Local or absolute max/min

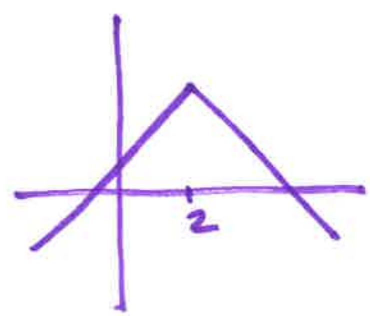
Q5 Absolute max : $f(4) = 5$
Absolute min : DNE
Local Max : $f(4) = 5$ and $f(6) = 4$
Local Min : $f(1) = 3$, $f(2) = 2$, $f(5) = 3$

Q6 Absolute max : DNE
Absolute min : $f(4) = 1$
Local Max : $f(3) = 4$ and $f(6) = 3$
Local Min : $f(2) = 2$ and $f(4) = 1$

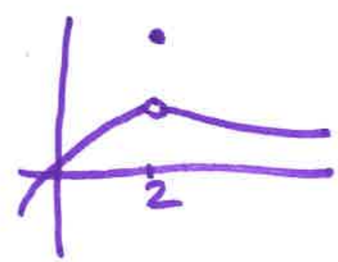
Q11 a)



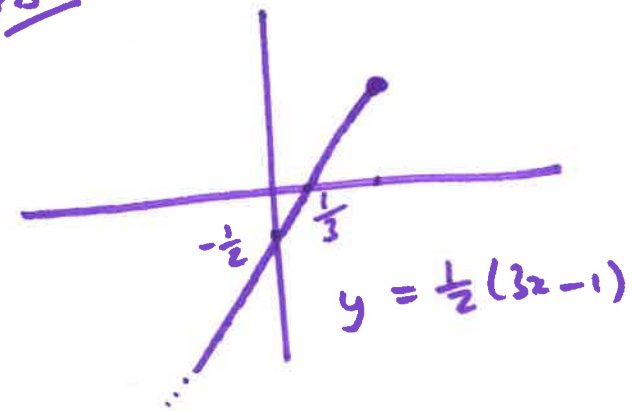
b)



c)



Q15

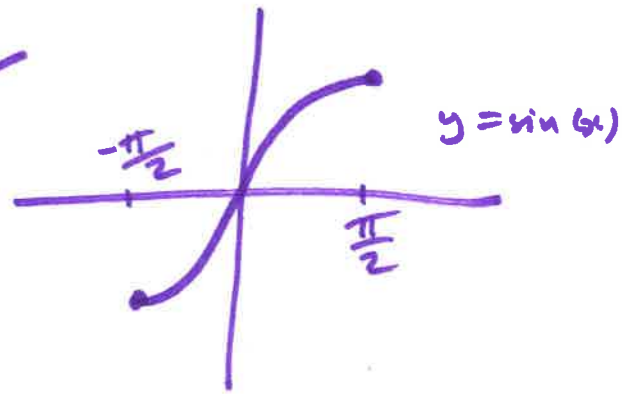


Absolute Max at $x=3$ $f(3) = 4$

No absolute min

No local max or mins.

Q21

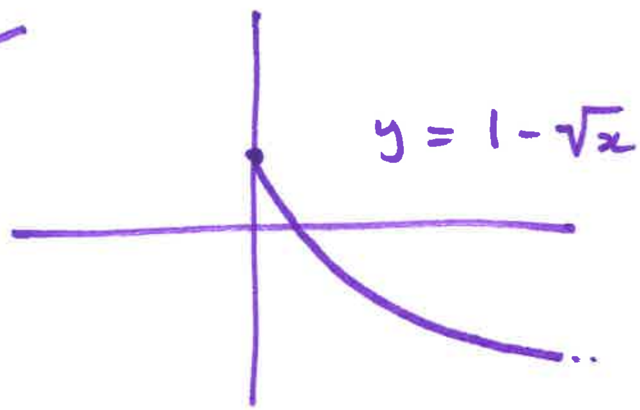


Absolute Max $= 1$

Absolute Min $= -1$

No local max or min.

Q25

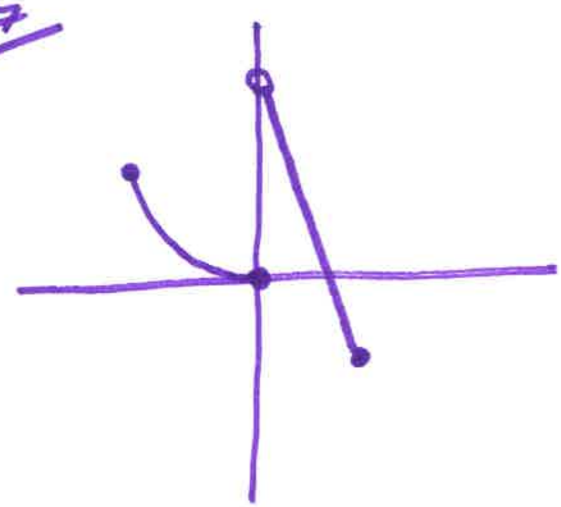


Absolute Max $= 1$

No Absolute Min

No local max/min.

Q27/



No Absolute Max

Absolute min $= -1$

Local min at 0

No local max.

Q30/ $f'(x) = 3x^2 + 6x - 15 = 3(x^2 + 2x - 3)$

$= 3(x+3)(x-1)$

$\Rightarrow$ $x = 1$ and -3 are only critical numbers

Q36/ $$h'(p) = \frac{1 \cdot (p^2+4) - 2p(p-1)}{(p^2+4)^2} = \frac{-p^2 + 2p + 4}{(p^2+4)^2}$$

$\Rightarrow$ $h'(p) = 0 \iff p^2 - 2p - 4 = 0$

$\iff p = 1 \pm \sqrt{5}$

These are the only critical numbers.

Q56/ $f'(t) = \dfrac{\frac{1}{2}t^{-\frac{1}{2}}(1+t^2) - 2t\sqrt{t}}{(1+t^2)^2}$

$f'(t) = 0 \iff \frac{1}{2}(1+t^2) - 2t^2 = 0$

$\iff 1 + t^2 - 4t^2 = 0 \iff t = \dfrac{1}{\pm\sqrt{3}}$

$f(0) = 0$

$f\left(\frac{1}{\sqrt{3}}\right) = \dfrac{\sqrt{\frac{1}{\sqrt{3}}}}{1+\frac{1}{3}} \approx \cancel{0.17534}\ 0.5698..$

$f(2) = \dfrac{\sqrt{2}}{5} \approx 0.2828..$

$\Rightarrow$ 0 is absolute min and $\dfrac{\sqrt{\frac{1}{\sqrt{3}}}}{1+\frac{1}{3}}$ is absolute max on $[0,2]$

Q62/ $f'(x) = 1 - \dfrac{2}{1+x^2} = \dfrac{x^2-1}{1+x^2} = 0 \iff x = \pm 1$

$f(0) = 0$

$f(1) = 1 - \frac{\pi}{2} \approx -0.5708$

$f(4) = 4 - 2\arctan(4) \approx 3.337$

$\Bigg\}$ Absolute min at 1
Absolute max at 4

Q77

$f(x)$ is a polynomial so is differentiable on all $\mathbb{R}$.

$f'(x) = 101x^{100} + 51x^{50} + 1 \geq 1$

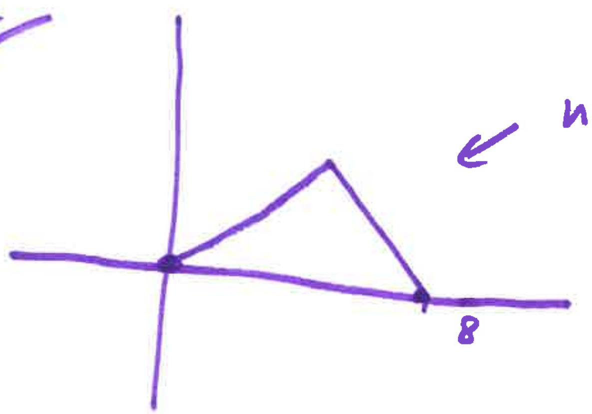
$\Rightarrow$ $f'(x) \neq 0$ for all x in $\mathbb{R}$

By Fermat c is a local max/min implies $f'(c) = 0$. We've shown this cannot occur. Hence there are no local max/mins.

§4.2

Q1 $f(0) = f(8)$ and $f(x)$ is continuous on $[0,8]$ and differentiable on $(0,8)$. If $c = 1$ (or $c = 5$) then $f'(c) = 0$

Q2



$\swarrow$ no point with $f'(c) = 0$

Q3/

a) Cts on $[0,8]$

Diff on $(0,8)$

b) $\dfrac{f(8)-f(0)}{8-0} = \dfrac{3}{8}$. Very roughly $f'(2) \approx \dfrac{3}{8}$

c) $\dfrac{f(6)-f(2)}{6-2} = \dfrac{-1}{2}$. Very roughly $f'(4) \approx \dfrac{-1}{2}$

Q7/ i) $\sin(\frac{x}{2})$ cts on $[\pi/2, \frac{3\pi}{2}]$

ii) $\sin(\frac{x}{2})$ diff. on $(\frac{\pi}{2}, \frac{3\pi}{2})$

$\sin(\frac{\pi}{4}) = \frac{1}{\sqrt{2}} = \sin(\frac{3\pi}{4})$

$f'(x) = -\frac{1}{2}\cos(\frac{x}{2}) = 0 \Rightarrow \frac{x}{2} = \frac{\pi}{2} + k\pi$ any integer.

$\Rightarrow x = \pi + 2k\pi$

Within $(\frac{\pi}{2}, \frac{3\pi}{2})$ only $c = \pi$ has the property that $f'(c) = 0$

Q10

$f'(x) = \sec^2(x) = \frac{1}{\cos^2(x)} \neq 0$ for any value of x.

Hence there is no c in $(0, \pi)$ with $f'(c) = 0$ even though $f(0) = f(\pi) = 0$. This does not contradict Rolle as $\tan(x)$ is not a continuous function on $[0, \pi]$. It is not even defined at $\frac{\pi}{2}$.

Q11

$f(0) = 1$, $f(2) = 3 \Rightarrow \frac{f(2) - f(0)}{2 - 0} = 1$

$f'(x) = 4x - 3 = 1 \Rightarrow x = 1$

$\Rightarrow c = 1$ is the only value in $(0, 2)$ with $f'(c) = \frac{f(2) - f(0)}{2 - 0}$.

Q12

$f(1) = \frac{1}{4}$ $f(4) = 1 \Rightarrow \frac{f(4) - f(1)}{4 - 1} = \frac{1 - \frac{1}{4}}{4 - 1} = \frac{1}{4}$

$f'(x) = \frac{-2}{(x-3)^3} = \frac{1}{4} \Rightarrow (x-3)^3 = -8 \Rightarrow x - 3 = -2 \Rightarrow x = 1$

But 1 is not in $(1, 4)$. So there is no c in $(1, 4)$ with $f'(c) = \frac{f(4) - f(1)}{4 - 1}$ ie $f(4) - f(1) = f'(c)(4 - 1)$

This does not contradict the M.V.T. as $\frac{1}{(x-3)^2}$ is <u>not</u> cts on $[1,4]$.

Q20/ $f(x) = x^3 + e^x$ cts. and diff. on $\mathbb{R}$

$$\left.\begin{matrix} f(-1) = -1 + \frac{1}{e} < 0 \\ f(1) = 1 + e > 0 \end{matrix}\right\} \begin{matrix}\text{There exists } c \text{ in } (-1,1) \text{ with} \\ f(c) = 0 \text{ by I.V.T.}\end{matrix}$$

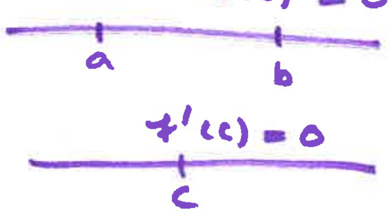
Assume there are at least 2 roots, ie $x_1 < x_2$ with $f(x_1) = f(x_2)$

Hence by Rolle's Theorem there exist c in (x_1, x_2) with $f'(c) = 0$

But $f'(c) = 3c^2 + e^c > 0$. Hence no such c can exist and there cannot be 2 distinct roots.

We conclude there is exactly one root.

Q24 a) $f(a) = f(b) = 0$



$f'(c) = 0$

$$\left.\begin{matrix} f(x) \\ f'(x) \end{matrix}\right\} \text{Rolles Theorem.}$$

b) $f(a) = f(b) = f(c) = 0$ f

a   b   c

$f'(d) = f'(e) = 0$ f'

d   e

$f''(g)$ f''

g

Rolle's Theorem 3 times

c) If $f(x)$ is differentiable n times on $\mathbb{R}$ and $f(x)$ has $n+1$ distinct roots then there exists c in $\mathbb{R}$ such that $f^{(n)}(c) = 0$

Q26 By M.V.T. can find c in $(2, 8)$ such that

$$f'(c) = \frac{f(8) - f(2)}{8-2}$$

$3 \leq f'(c) \leq 5$ $\Rightarrow 3 \leq \frac{f(8) - f(2)}{8-2} \leq 5$ $\Rightarrow 18 \leq f(8) - f(2) \leq 30$

Q33

$$f'(x) = \frac{-1}{x^2}$$

$$g'(x) = \begin{cases} \frac{-1}{x^2} & x > 0 \\ \frac{-1}{x^2} & x < 0 \end{cases} \Rightarrow f'(x) = g'(x) \text{ on all } \mathbb{R} \text{ minus } 0$$

However $f(x) - g(x) = \begin{cases} 0 & x > 0 \\ -1 & x < 0 \end{cases}$ $\Leftarrow$ <u>not</u> a constant function on $\mathbb{R}$ minus 0.

Corollary 7 does <u>not</u> apply in this case as $f'(x) = g'(x)$ on an <u>open interval</u>. $\mathbb{R}$ minus 0 is <u>not</u> an open interval.