Homework 7 Solutions

§3.5

Q5/ $x^2 - 4xy + y^2 = 4 \Rightarrow 2x - 4y - 4x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$

$\Rightarrow \frac{dy}{dx} = \frac{2x-4y}{4x-2y}$

Q9/ $\frac{x^2}{x+y} = y^2+1 \Rightarrow x^2 = xy^2 + y^3 + x + y$

$\Rightarrow 2x = y^2 + 2xy\frac{dy}{dx} + 3y^2\frac{dy}{dx} + 1 + \frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{2x - y^2 - 1}{3y^2 + 2xy + 1}$

Q14/ $e^y \sin(x) = x + xy \Rightarrow \frac{dy}{dx} \cdot e^y \sin(x) + e^y \cos(x) = 1 + y + x\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{1 + y - e^y\cos(x)}{e^y \sin(x) - x}$

Q17/ $\arctan(x^2 y) = x + xy^2 \Rightarrow x^2 y = \tan(x + xy^2) \Rightarrow$

$2xy + x^2\frac{dy}{dx} = \sec^2(x+xy^2)\left(1 + y^2 + 2xy\frac{dy}{dx}\right)$

$$\Rightarrow \frac{dy}{dx} = \frac{2xy - \sec^2(x+xy^2)(1+y^2)}{\sec^2(x+xy^2)\cdot 2xy - x^2}$$

Q20/ $\tan(x-y) = \dfrac{y}{1+x^2} \Rightarrow \sec^2(x-y)\left(1 - \dfrac{dy}{dx}\right) = \dfrac{\frac{dy}{dx}(1+x^2) - 2xy}{(1+x^2)^2}$

$$\Rightarrow \frac{dy}{dx} = \frac{\sec^2(x-y) + \frac{2xy}{(1+x^2)^2}}{\sec^2(x-y) + \frac{1}{(1+x^2)}}$$

Q22/ $g(x) + x\sin(g(x)) = x^2 \Rightarrow g'(x) + \sin(g(x)) + x\cos(g(x))g'(x) = 2x$

$$\Rightarrow g'(x) = \frac{2x - \sin(g(x))}{1 + x\cos(g(x))}$$

Note that $g(0) + 0\sin(g(0)) = 0^2 \Rightarrow g(0) = 0$

$$\Rightarrow g'(0) = \frac{2\cdot 0 - \sin(0)}{1 + x\cos(0)} = 0.$$

Q24

$$y\sec(x) = x\tan(y)$$

$$\Rightarrow \sec(x) + y\tan(x)\sec(x)\frac{dx}{dy} = \frac{dx}{dy}\tan(y) + x\sec^2(y)$$

$$\Rightarrow \frac{dx}{dy} = \frac{x\sec^2(y) - \sec(x)}{y\tan(x)\sec(x) - \tan(y)}$$

Q24

$$x^2 + y^2 = (2x^2 + 2y^2 - x)^2$$

$$\Rightarrow 2x + 2y\frac{dy}{dx} = 2(2x^2+2y^2-x)\cdot\left(4x + 4y\frac{dy}{dx} - 1\right)$$

$$\Rightarrow 2\cdot 0 + 2\cdot\tfrac{1}{2}\cdot\frac{dy}{dx}\Big|_{\substack{x=0\\y=\frac{1}{2}}} = 2\left(2\cdot 0^2 + 2\cdot\tfrac{1}{2}^2 - 0\right)\cdot\left(4\cdot 0 + 4\cdot\tfrac{1}{2}\frac{dy}{dx}\Big|_{\substack{x=0\\y=\frac{1}{2}}} - 1\right)$$

$$\Rightarrow \frac{dy}{dx}\Big|_{\substack{x=0\\y=\frac{1}{2}}} = \cancel{4}\,2\,\frac{dy}{dx}\Big|_{\substack{x=0\\y=\frac{1}{2}}} - \cancel{2}\,1 \Rightarrow \frac{dy}{dx}\Big|_{\substack{x=0\\y=\frac{1}{2}}} = 1$$

$\Rightarrow$ Tangent has equation $y - \frac{1}{2} = 1.(x-0) \Rightarrow y = x + \frac{1}{2}$.

Q32

$$y^4 - 4y^2 = x^4 - 5x^2 \Rightarrow (4y^3 - 8y)\frac{dy}{dx} = 4x^3 - 10x$$

$\Rightarrow \frac{dy}{dx} = \frac{4x^3 - 10x}{4y^3 - 8y} \Rightarrow \frac{dy}{dx}\Big|_{\substack{x=0\\y=-2}} = 0 \Rightarrow y = -2$ is tangent at $(0,-2)$.

Q39 $xy + e^y = e \Rightarrow y + x\frac{dy}{dx} + e^y \frac{dy}{dx} = 0$

$$\Rightarrow \frac{dy}{dx} = \frac{-y}{x + e^y}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-\frac{dy}{dx}(x+e^y) - \left(1 + e^y\frac{dy}{dx}\right)(-y)}{(x+e^y)^2}$$

$$x = 0 \Rightarrow y = 1 \Rightarrow \left.\frac{dy}{dx}\right|_{\substack{x=0\\y=1}} = \frac{-1}{e} \Rightarrow \left.\frac{d^2y}{dx^2}\right|_{\substack{x=0\\y=1}} = \frac{-\left(\frac{-1}{e}\right)\cdot e - \left(1+e\left(\frac{-1}{e}\right)\right)(-1)}{e^2}$$

$$= \frac{1}{e^2}.$$

Q75 $x^2y^2 + xy = 2 \Rightarrow 2xy^2 + 2x^2y\frac{dy}{dx} + y + x\frac{dy}{dx} = 0$

$$\Rightarrow \frac{dy}{dx} = \frac{-2xy^2 - y}{2x^2y + x} = \frac{-y}{x} = -1 \Rightarrow x = y$$

$$\Rightarrow x^4 + x^2 = 2 \Rightarrow (x^2)^2 + (x^2) - 2 = 0 \Rightarrow x^2 = \frac{-1 \pm \sqrt{9}}{2} = -2 \text{ or } 1$$

$\Rightarrow x = \pm 1 \Rightarrow (1,1)$ and $(-1,-1)$ are points whose tangent has slope -1.

§3.6

Q5 $f(x) = \ln\left(\frac{1}{x}\right) = -\ln(x) \Rightarrow f'(x) = \frac{-1}{x}$.

Q12 $h'(x) = \dfrac{1 + \frac{1}{2}(x^2-1)^{-\frac{1}{2}} \cdot 2x}{x + \sqrt{x^2-1}}$

Q21 $\dfrac{dy}{dx} = \sec^2(\ln(ax+b)) \cdot \dfrac{a}{ax+b}$

Q25 $y' = \dfrac{\tan(x)\sec(x)}{\sec(x)} = \tan(x) \Rightarrow y'' = \sec^2(x)$

Q30 $f(x) = \ln(\underbrace{\ln(\overbrace{\ln(x)}^{>1})}_{>0}) \Rightarrow \ln(x) > 1$ then $x > e \Rightarrow$ Domain is (e, ∞)

$f'(x) = \dfrac{\left(\dfrac{\frac{1}{x}}{\ln(x)}\right)}{\ln(\ln(x))} = \dfrac{1}{x\ln(x)\ln\ln(x)}$.

Q33 $\left.\dfrac{dy}{dx}\right|_{x=3} = \left.\dfrac{2x-3}{x^2-3x+1}\right|_{x=3} = \dfrac{3}{1} = 3 \Rightarrow$ Tangent has equation $y = 3(x-3)$.

Q39 $\ln(y) = 2\ln(x^2+2) + 4\ln(x^4+4)$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{4x}{x^2+2} + \frac{16x^3}{x^4+4}$

$\Rightarrow \frac{dy}{dx} = (x^2+2)^2(x^4+4)^4 \cdot \left(\frac{4x}{x^2+2} + \frac{16x^3}{x^4+4}\right).$

Q49 $y = \tan(x)^{\frac{1}{x}} \Rightarrow \ln(y) = \frac{\ln(\tan(x))}{x}$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{\frac{\sec^2(x)}{\tan(x)} \cdot x - \ln(\tan(x))}{x^2}$

$\Rightarrow \frac{dy}{dx} = \tan(x)^{\frac{1}{x}} \cdot \left(\frac{\frac{\sec^2(x)}{\tan(x)}\, x - \ln(\tan(x))}{x^2}\right)$

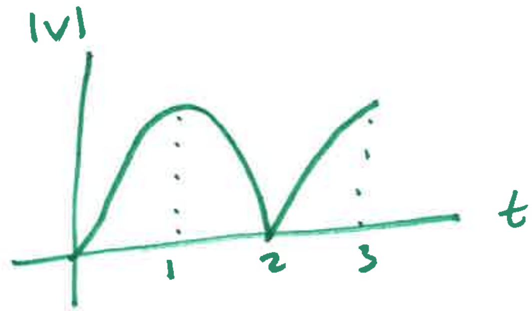
Q52 $x^y = y^x \Rightarrow y\ln(x) = x\ln(y)$

$\Rightarrow \frac{dy}{dx}\ln(x) + \frac{y}{x} = \ln(y) + \frac{x}{y}\frac{dy}{dx}$

$\Rightarrow \frac{dy}{dx} = \frac{\ln(y) - \frac{y}{x}}{\ln(x) - \frac{x}{y}}.$

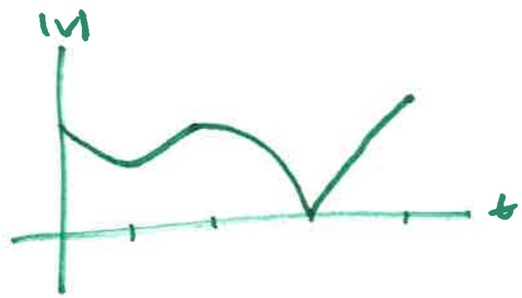
§3.7

Q5/ a)



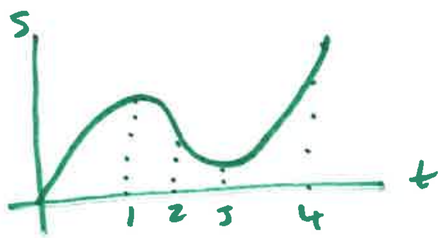
$\Rightarrow$ Speeding up on [0, 1] and [2, 3]
slowing down on [1, 2]

a)

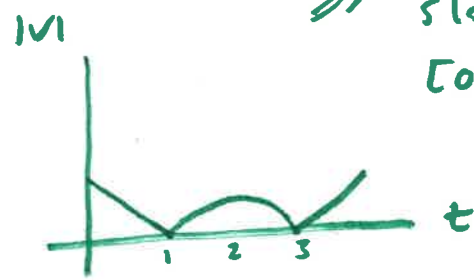


$\Rightarrow$ speeding up on [1, 2] and [3, 4]
slowing down on [0, 1] and [2, 3]

Q6/ a)

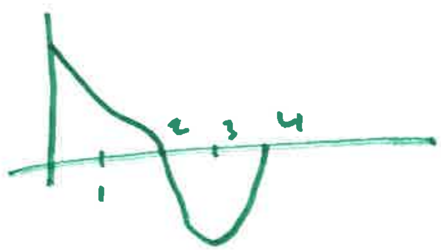


$\longrightarrow$

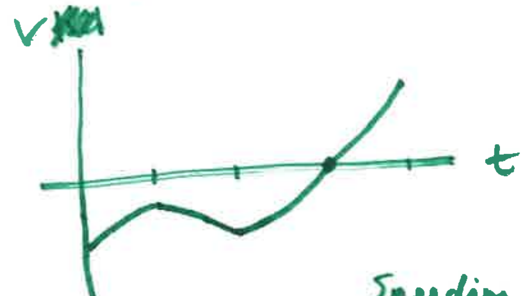


$\Rightarrow$ Speeding up on [1,2] and [3, 4]
slowing down on [0,1] and [2,3]

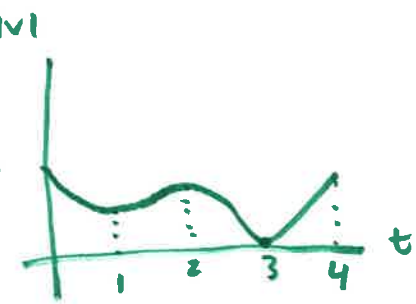
b)



$\longrightarrow$



$\Rightarrow$



$\Rightarrow$ Speeding up on [1,2] and [3,4]
slowing down on [0, 1] and [2, 3]

Q7

a) $\frac{dh}{dt} = 24.5 - 9.8t \Rightarrow v(2) = 24.5 - (9.8)\cdot 2 = 4.9 \text{ m/s}$

$v(4) = 24.5 - (9.8)\cdot 4 = -14.7 \text{ m/s}$

b) $v(t) = 0 \Rightarrow 24.5 - 9.8t = 0 \Rightarrow t = 2.5$ s.

c) $h(2.5) = 32\frac{5}{8}$ m

d) $h(t) = 0 \Rightarrow 2 + 24.5t - 4.9t^2 = 0$

$\Rightarrow t = \frac{-24.5 - \sqrt{(24.5)^2 + 8\cdot(4.9)}}{-2\cdot(4.9)} \approx 5.08$ s

$t \geq 0$

e) $v(5.08) \approx -25.3$ m/s.

Q14

$\frac{dr}{dt} = 60$ cm/s and $r(0) = 0$

$\Rightarrow r(1) = 60,\ r(2) = 120,\ r(3) = 180,\ r(4) = 240,\ r(5) = 300$

$A = \pi r^2 \Rightarrow \frac{dA}{dt} = \frac{d\pi r^2}{dt} = \pi \frac{dr^2}{dt} = \pi \cdot 2r \cdot \frac{dr}{dt} = 120\pi r$

$\Rightarrow \left.\frac{dA}{dt}\right|_{t=1} = 120\pi\cdot 60,\ \left.\frac{dA}{dt}\right|_{t=3} = 120\cdot\pi\cdot 180,\ \left.\frac{dA}{dt}\right|_{t=5} = 120\cdot\pi\cdot 300$

Conclusion: The rate of change of A with respect to time is increasing.

Q18

$$\frac{dV}{dt} = 5000 \cdot 2 \cdot \left(1 - \frac{1}{40}t\right) \cdot \left(\frac{-1}{40}\right)$$

$$= \frac{-10000}{40}\left(1 - \frac{1}{40}t\right)$$

a), b), c), d) just feed $t = 5, 10, 20, 40$ into this function.

For t in $[0, 40]$ $\quad 1 - \frac{1}{40}t \geq 0$

$\Rightarrow \quad \left|\frac{dV}{dt}\right| = \frac{10000}{40}\left(1 - \frac{1}{40}t\right) \Leftarrow$ straight line with negative slope, hence decreasing.

$\Rightarrow$ The water is flowing out the fastest at $t = 5$ mins.

Q24 $\quad [C](t) = \frac{a^2kt}{1 + akt}$, $\quad [A](0) = [B](0) = a$

a) $$\frac{d[C]}{dt} = \frac{a^2k(1 + akt) - ak(a^2kt)}{(1 + akt)^2} = \frac{a^2k}{(1 + akt)^2}$$

b) $$k(a - [C](t))^2 = k\left(a - \frac{a^2kt}{1 + akt}\right)^2 = k\left(\frac{a + \cancel{a^2kt} - \cancel{a^2kt}}{1 + akt}\right)^2$$

$$= \frac{a^2k}{(1 + akt)^2} = \frac{d[C]}{dt}$$

c) $\lim_{t\to\infty} [C](t) = \frac{a^2k}{ak} = a$

d) $\lim_{t\to\infty} \frac{d[C]}{dt} = 0$

e) Over time the reaction will completely stop and the concentration of C will be a.

Q26/ $f(0) = 20 \Rightarrow 20 = \frac{a}{1+b}$

$f'(t) = \frac{-a}{(1+be^{-0.7t})^2} \cdot -0.7 \cdot be^{-0.7t}$

$\Rightarrow f'(0) = \frac{0.7ab}{(1+b)^2} = 12$

$\Rightarrow 0.7 \cdot \frac{a}{1+b} \cdot \frac{b}{1+b} = 12 \Rightarrow (0.7) \cdot 20 \cdot \frac{b}{1+b} = 12$

$\Rightarrow \frac{b}{1+b} = \frac{12}{14} \Rightarrow 14b = 12b + 12 \Rightarrow b = 6 \Rightarrow a = 140$

$\lim_{t\to\infty} f(t) = a = 140$. Population will approach 140 in long run.