Homework 6 Solutions

§3.1

Q12 $\frac{dB}{dy} = -6cy^{-7}$

Q23 $y = x^{3/2} + 4x^{1/2} + 3x^{-\frac{1}{2}} \Rightarrow \frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} + 2x^{-\frac{1}{2}} - \frac{3}{2}x^{-3/2}$

Q27 $G(q) = (1+q^{-1}) = 1 + 2q^{-1} + q^{-2} \Rightarrow \frac{dG}{dq} = -2q^{-2} - 2q^{-3}$

Q33 $\frac{dy}{dx} = 6x^2 - 2x \Rightarrow \left.\frac{dy}{dx}\right|_{x=1} = 4 \Rightarrow$ Tangent has equation

$y-3 = 4(x-1)$.

Q37 $\frac{dy}{dx} = 4x^3 + 2e^x \Rightarrow \left.\frac{dy}{dx}\right|_{x=0} = 2$

$\Rightarrow$ Tangent has equation $(y-2) = 2(x-0)$

Normal has equation $(y-2) = \frac{-1}{2}(x-0)$

Q49 a) $s = t^3 - 3t \Rightarrow v = \frac{ds}{dt} = 3t^2 - 3 \Rightarrow a = \frac{dv}{dt} = 6t$

b) $a(2) = 12$ meters per second per second

c) $v = 0 \Leftrightarrow 3t^2 - 3 = 0 \Leftrightarrow t = \pm 1$

Hence when $v=0$ $a = \pm 6$.

Q56 $f'(x) = e^x - 2 = 0 \Leftrightarrow x = \ln(2)$ ~~[illegible]~~

Q75

$y = ax^2 + bx \Rightarrow \frac{dy}{dx} = 2ax + b$

Need a and b such that $1 = a + b$ ($(1,1)$ must be on curve)

and $2a + b = 3$ (slope of tangent must be 3)

$\Rightarrow a = 2 \Rightarrow b = -1$.

Q81 Both x^2 and $mx + b$ are differentiable on all $\mathbb{R}$.

$\Rightarrow f(x)$ certainly differentiable at $x \neq 2$

Must first be continuous at $x = 2$.

$\lim_{x \to 2^+} f(x) = 2m + b$ and $\lim_{x \to 2^-} f(x) = 2^2 = 4 = f(2)$

$\Rightarrow$ Need $2m + b = 4$

Now require differentiability :

$\lim_{h \to 0^+} \frac{f(2+h) - f(2)}{h} = m$ and $\lim_{h \to 0^-} \frac{f(2+h) - f(2)}{h} = 2.2 = 4$

$\Rightarrow$ Need $m = 4 \Rightarrow b = -4$.

§3.2

Q13 $\dfrac{dy}{dz} = \dfrac{2x(x^3-1) - 3x^2(x^2+1)}{(x^3-1)^2}$

Q26 $f'(x) = \dfrac{a(cx+d) - c(ax+b)}{(cx+d)^2} = \dfrac{(ad-bc)}{(cx+d)^2}$

Q28 $f(x) = \sqrt{x}\, e^x \Rightarrow f'(x) = \frac{1}{2}x^{-\frac{1}{2}}e^x + x^{\frac{1}{2}}e^x$

$= \left(\frac{1}{2}x^{-\frac{1}{2}} + x^{\frac{1}{2}}\right)e^x$

$\Rightarrow f''(x) = \left(\frac{-1}{4}x^{-\frac{3}{2}} + \frac{1}{2}x^{-\frac{1}{2}}\right)e^x + \left(\frac{1}{2}x^{-\frac{1}{2}} + x^{\frac{1}{2}}\right)e^x$

Q42 $g(x) = x/e^x \Rightarrow g'(x) = \dfrac{e^x - xe^x}{(e^x)^2} = \dfrac{1-x}{e^x}$

$\Rightarrow g''(x) = \dfrac{-1\cdot e^x - (1-x)e^x}{(e^x)^2} = \dfrac{x-2}{e^x}$

$\Rightarrow g^{(3)}(x) = \dfrac{e^x - (x-2)e^x}{(e^x)^2} = \dfrac{3-x}{e^x} \quad \dots \Rightarrow g^{(n)}(x) = \begin{cases} \dfrac{x-n}{e^x} & \text{if } n \text{ even} \\ \dfrac{n-x}{e^x} & \text{if } n \text{ odd} \end{cases}$

$\Rightarrow g^{(n)}(x) = (-1)^n \cdot \dfrac{x-n}{e^x}$.

Q48

$$f'(x) = x^2 f(x) \Rightarrow f''(x) = 2x f(x) + x^2 f'(x)$$

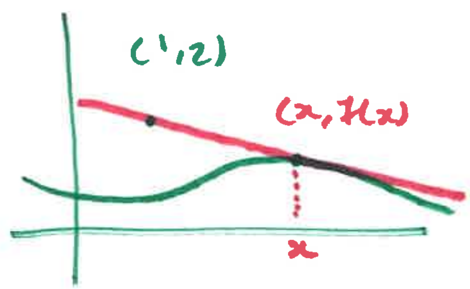
$$= 2x f(x) + x^4 f(x)$$

$$\Rightarrow f''(2) = 4 f(2) + 16 f(2) = 200.$$

Q51 a) $\dfrac{dy}{dx} = g(x) + x g'(x)$, b) $\dfrac{dy}{dx} = \dfrac{g(x) - x g'(x)}{(g(x))^2}$

c) $\dfrac{dy}{dx} = \dfrac{g'(x) - g(x)}{x^2}$

Q53 $y = \dfrac{x}{x+1} \Rightarrow \dfrac{dy}{dx} = \dfrac{1\cdot(x+1) - x\cdot 1}{(x+1)^2} = \dfrac{1}{(x+1)^2}$



$y = f(x)$ $\Rightarrow$ Need $f'(x) = \dfrac{f(x) - 2}{x - 1}$

(Vague picture of method)

In our case $\dfrac{1}{(x+1)^2} = \dfrac{\frac{x}{x+1} - 2}{x - 1} \Rightarrow x - 1 = x(x+1) - 2(x+1)^2$

$$\Rightarrow x^2 + 4x + 1 = 0 \Rightarrow x = -2 \pm \frac{\sqrt{12}}{2} = -2 \pm \sqrt{3}$$

$\Rightarrow$ There are 2 tangents and they touch the curve at

$$\left(-2+\sqrt{3}, \frac{-2+\sqrt{3}}{-1+\sqrt{3}}\right) \text{ and } \left(-2-\sqrt{3}, \frac{-2-\sqrt{3}}{-1-\sqrt{3}}\right).$$

§3.3

Q8/ $f(t) = \dfrac{\cot(t)}{e^t} = (\tan t\, e^t)^{-1} \Rightarrow f'(t) = -1 \cdot (\tan(t) \cdot e^t)^{-2} \cdot (\sec^2(t)\, e^t + \tan(t)\, e^t)$

Q15/ $f'(\theta) = \cos\theta \sin\theta + \theta(-\sin\theta)\sin\theta + \theta\cos\theta\cos\theta$

Q30/ $f(t) = \sec(t) \Rightarrow f'(t) = \tan(t)\sec(t) = \dfrac{\sin(t)}{\cos^2(t)}$

$$\Rightarrow f''(t) = \frac{\cos^3(t) - \sin(t) \cdot 2\cos(t) \cdot (-\sin(t))}{\cos^4(t)}$$

$$\Rightarrow f''\left(\frac{\pi}{4}\right) = \frac{\left(\frac{1}{\sqrt{2}}\right)^3 + 2\left(\frac{1}{\sqrt{2}}\right)^3}{\left(\frac{1}{\sqrt{2}}\right)^4} = \frac{3}{\frac{1}{\sqrt{2}}} = 3\sqrt{2}.$$

Q31/ a) $f(x) = \dfrac{\tan(x) - 1}{\sec(x)} \Rightarrow f'(x) = \dfrac{\sec^2(x) \cdot \sec(x) - \tan(x)\sec(x)(\tan(x) - 1)}{\sec^2(x)}$

$\Rightarrow f'(x) = \dfrac{1 + \tan(x)}{\sec(x)}$ $\quad (1 + \tan^2(x) = \sec^2(x))$

b) $$F(x) = \frac{\dfrac{\sin(x)}{\cos(x)} - 1}{\dfrac{1}{\cos(x)}} = \sin(x) - \cos(x) \Rightarrow F'(x) = \cos(x) + \sin(x)$$

c) $$\frac{1 + \tan(x)}{\sec(x)} = \frac{1 + \dfrac{\sin(x)}{\cos(x)}}{\dfrac{1}{\cos(x)}} = \cos(x) + \sin(x)$$

Q44/ $$\lim_{x \to 0} \frac{\sin(3x)\sin(5x)}{x^2} = \lim_{x \to 0} \left(3 \cdot 5 \cdot \frac{\sin(3x)}{3x} \cdot \frac{\sin(5x)}{5x} \right)$$

But $$\lim_{x \to 0} \frac{\sin(3x)}{3x} = 1 = \lim_{x \to 0} \frac{\sin(5x)}{5x}$$

$$\Rightarrow \lim_{x \to 0} \frac{\sin(3x)\sin(5x)}{x^2} = 15.$$

Q48/ $$\lim_{u \to 0} \frac{\sin(u)}{u} = 1, \quad \lim_{x \to 0} x^2 = 0 \Rightarrow \lim_{x \to 0} \frac{\sin(x^2)}{x^2} = 1$$

$$\frac{\sin(x^2)}{x} = \frac{\sin(x^2)}{x^2} \cdot x \Rightarrow \lim_{x \to 0} \frac{\sin(x^2)}{x} = \lim_{x \to 0} \frac{\sin(x^2)}{x^2} \cdot \lim_{x \to 0} x = 1 \cdot 0$$
$$= 0.$$

§3.4

Q17 $f'(x) = 4(2x-3)^3 \cdot 2 \cdot (x^2+x+1)^5 + (2x-3)^4 \cdot 5 \cdot (x^2+x+1)^4 \cdot (2x+1)$

Q21 $\frac{dy}{dx} = \frac{1}{2}\left(\frac{x}{x+1}\right)^{-\frac{1}{2}} \cdot \frac{1\cdot(x+1) - 1\cdot x}{(x+1)^2}.$

Q30 $J'(\theta) = 2\tan(n\theta) \cdot \sec^2(n\theta) \cdot n.$

Q41 $f'(t) = 2\sin(e^{\sin^2(t)}) \cdot \cos(e^{\sin^2(t)}) \cdot e^{\sin^2(t)} \cdot 2\sin(t)\cos(t)$

Q44 $\frac{dy}{dx} = \ln(2) \cdot 2^{3^{4^x}} \cdot \ln(3)\, 3^{4^x} \cdot \ln(4) \cdot 4^x$

Q48 $\frac{dy}{dx} = -2(1+\tan(x))^{-3}\sec^2(x)$

$\Rightarrow \frac{d^2y}{dx^2} = 6(1+\tan(x))^{-4} \cdot \sec^4(x) - 2(1+\tan(x))^{-3} \cdot 2\sec^2(x)\tan(x)$

Q60 $6x+2y=1 \Rightarrow y = 1-3x \Rightarrow$ Need x such that derivative is $\frac{1}{3}$.

$\frac{d}{dx}(1+2x)^{\frac{1}{2}} = (1+2x)^{-\frac{1}{2}} = \frac{1}{3} \Rightarrow 1+2x = 3^2 \Rightarrow x = 4.$

Q62 $h'(x) = \frac{1}{2}\cdot(4+3f(x))^{-\frac{1}{2}} \cdot 3f'(x) \Rightarrow h'(1) = \frac{1}{2}\cdot(4+3.7)^{-\frac{1}{2}} \cdot 3 \cdot 4$

$\Rightarrow h'(1) = \frac{6}{5}.$

Q67

$$g'(x) = \frac{1}{2} f(x)^{-\frac{1}{2}} \cdot f'(x)$$

$$\Rightarrow g'(3) = \frac{1}{2} f(3)^{-\frac{1}{2}} \cdot f'(3) = \frac{1}{2}\, 2^{-\frac{1}{2}} \cdot -\frac{2}{3} = \frac{-1}{3\sqrt{2}}$$

Q78 $f(x) = \frac{x}{e^x}$ and $f^{(n)}(x) = (-1)^n \frac{x-n}{e^x}$ by Q42 in §3.2

$$\Rightarrow f^{(1000)}(x) = \frac{x-1000}{e^x}$$

Q97 Let $\sin_d(\theta)$ = sin function in degrees. Same for $\cos_d(\theta)$

Degrees $\longleftrightarrow$ Radians

$$\theta \longleftrightarrow \frac{2\pi}{360}\theta = \frac{\pi}{180}\theta$$

$$\Rightarrow \sin_d(\theta) = \sin\left(\frac{\pi}{180}\theta\right) \text{ and } \cos_d(\theta) = \cos\left(\frac{\pi}{180}\theta\right)$$

usual sine function (always in radians)

$$\Rightarrow \frac{d}{d\theta}\sin_d(\theta) = \frac{d \sin\left(\frac{\pi}{180}\theta\right)}{d\theta} = \frac{\pi}{180}\cos\left(\frac{\pi}{180}\theta\right) = \frac{\pi}{180}\cos_d(\theta).$$