Math 1A Homework 4 Solutions

§2.4

Q1/ $\delta = 0.1$ (or smaller)

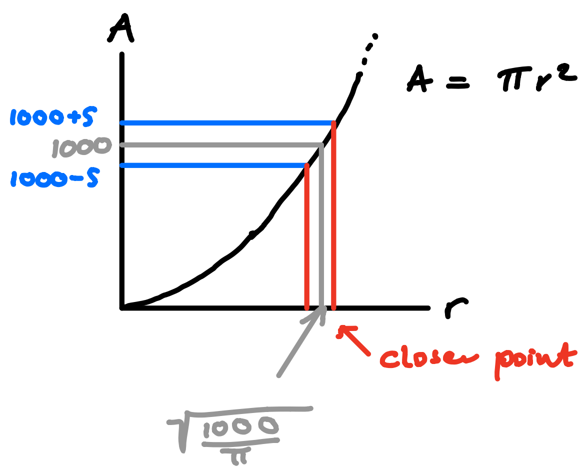
Q2/ $\delta = 0.4$ (or smaller)

Q3/ $\delta = 4 - (1.6)^2 = 1.44$ (or smaller)

Q11/

a) $\pi r^2 = 1000 \Rightarrow r = \sqrt{\frac{1000}{\pi}}$

b) $\varepsilon = 5$



$\Rightarrow \delta = \sqrt{\frac{1005}{\pi}} - \sqrt{\frac{1000}{\pi}}$

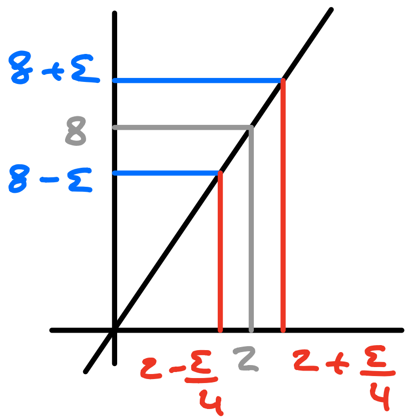
≈ 0.0445 cm

$\Rightarrow$ The radius must be within 0.0445 cm to ensure area is between 995 cm^2 and 1005 cm^2

c) $f(x) = \pi x^2$, $a = \sqrt{\frac{1000}{\pi}}$, $L = 1000$, $\varepsilon = 5$,

$\delta = 0.0445$

13/ Fix $\varepsilon > 0$



$\Rightarrow$ Choose $\delta = \frac{\varepsilon}{4}$

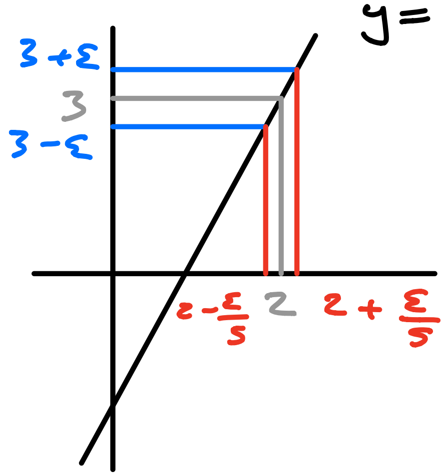
Verify :

$$|x-2| < \frac{\varepsilon}{4} \Rightarrow 4|x-2| < \varepsilon \Rightarrow |4x-8| < \varepsilon$$

a) For $\varepsilon = 0.1$, choose $\delta = \frac{0.1}{4} = 0.025$

b) For $\varepsilon = 0.01$, choose $\delta = \frac{0.01}{4} = 0.0025$

14/ Fix $\varepsilon > 0$



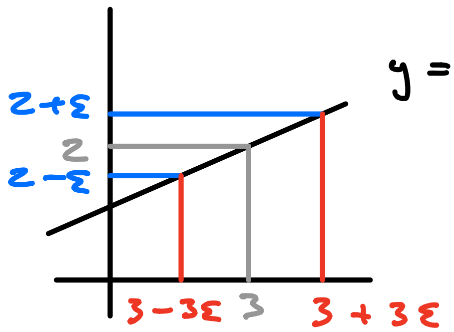
$\Rightarrow$ Choose $\delta = \frac{\varepsilon}{5}$

a) $\varepsilon = 0.1$, choose $\delta = \frac{0.1}{5}$

b) $\varepsilon = 0.05$, choose $\delta = \frac{0.05}{5}$

c) $\varepsilon = 0.01$, choose $\delta = \frac{0.01}{5}$

Q15 Fix $\varepsilon > 0$



$y = 1 + \frac{1}{3}x$

$\Rightarrow$ Choose $\delta = 3\varepsilon$

Verify :

$0 < |x-3| < 3\varepsilon \Rightarrow \frac{1}{3}|x-3| < \varepsilon \Rightarrow \left|\frac{1}{3}x - 1\right| < \varepsilon$

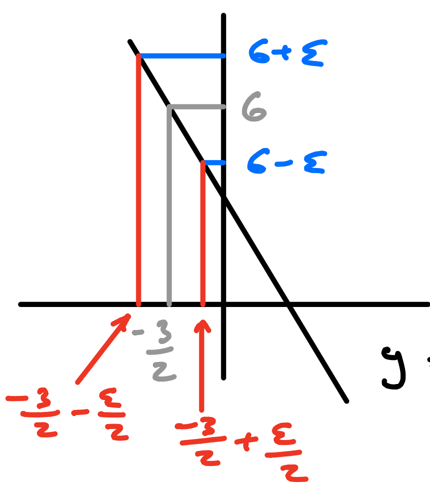
$\Rightarrow \left|(1+\frac{1}{3}x) - 2\right| < \varepsilon$

Q22 $\frac{9-4x^2}{3+2x} = \frac{(3+2x)(3-2x)}{(3+2x)} = 3-2x$ $x \neq -\frac{3}{2}$

$\Rightarrow$

$\lim_{x \to -\frac{3}{2}} \frac{9-4x^2}{3+2x} = \lim_{x \to -\frac{3}{2}} 3-2x$

Fix $\varepsilon > 0$



$y = 3-2x$

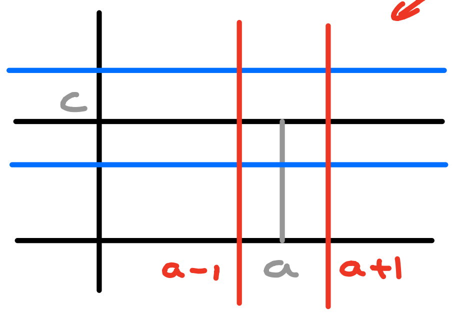
$\Rightarrow$ Choose $\delta = \frac{\varepsilon}{2}$

$0 < \left|x - (-\frac{3}{2})\right| < \frac{\varepsilon}{2} \Rightarrow 2\left|x + \frac{3}{2}\right| < \varepsilon \Rightarrow |2x+3| < \varepsilon$

$\Rightarrow |-2x-3| < \varepsilon \Rightarrow |(3-2x)-6| < \varepsilon$

24/ Fix $\varepsilon > 0$

Any vertical strip works



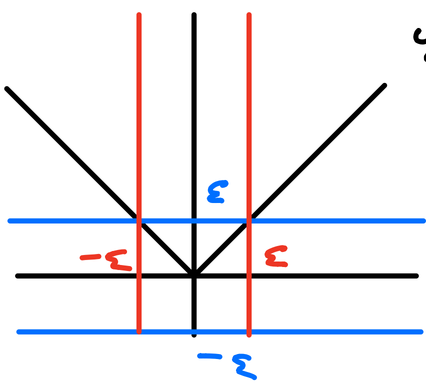
$y = c$

$\Rightarrow$ Let $\delta = 1$ for example

Always true

$$0 < |x-a| < 1 \Rightarrow |c-c| = 0 < \varepsilon$$

25/ Fix $\varepsilon > 0$



$y = |x|$

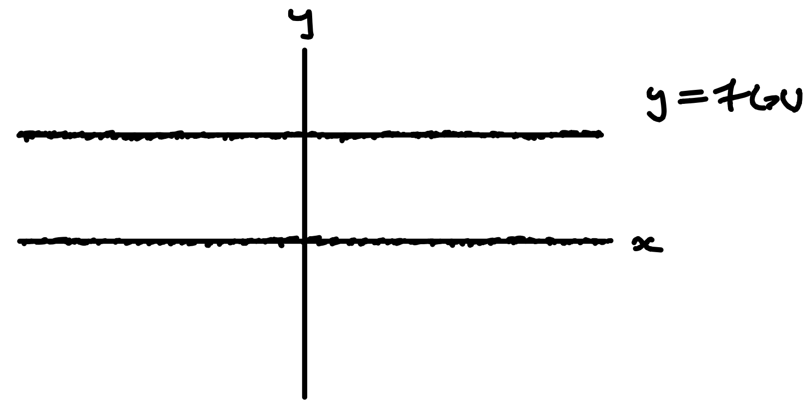
$\Rightarrow$ Choose $\delta = \varepsilon$

Verify :

$$0 < |x-0| < \varepsilon \Rightarrow |x| < \varepsilon \Rightarrow \big||x|-0\big| < \varepsilon$$

39/ (Hard)

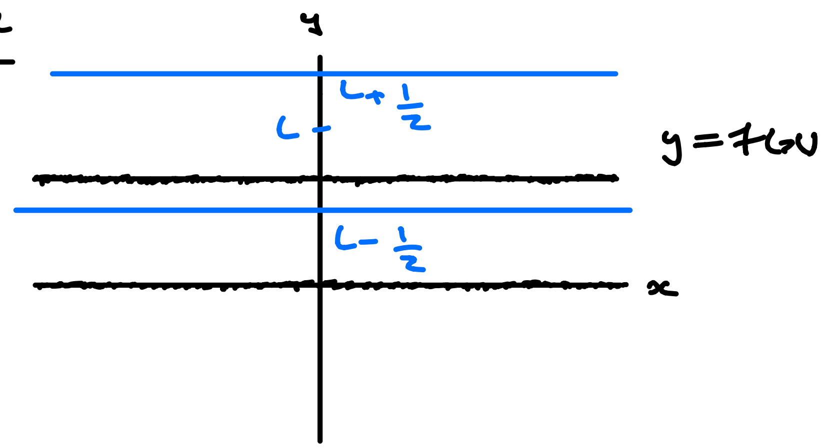
The rational and irrational numbers are freely mixed on $\mathbb{R}$. This means $y = f(x)$ looks is two horizontal lines (with holes) at $y = 1$ and $y = 0$



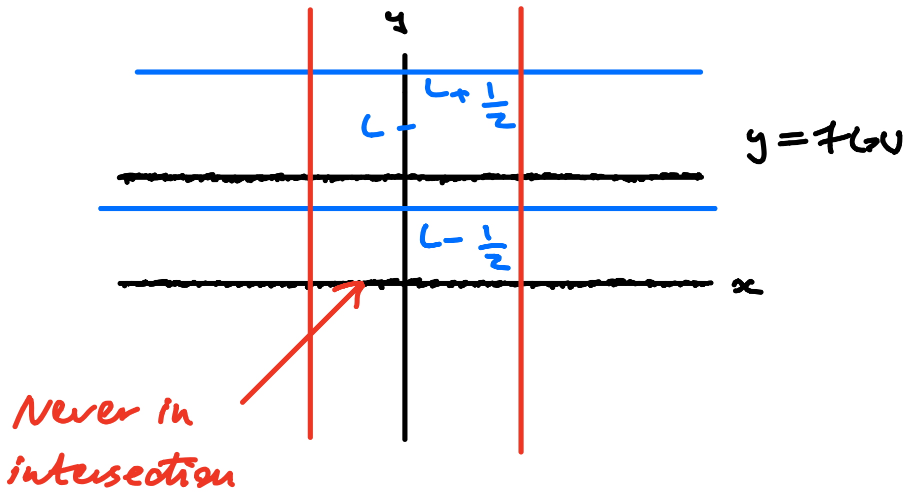
We need to prove $\lim_{x \to 0} f(x) \neq L$ for any L.

Observation : regardless of L, if $\varepsilon = \frac{1}{2}$, no horizontal strip can completely contain all $y = f(x)$

For example



Hence for $\varepsilon = \frac{1}{2}$ there is no vertical strip such that $y = f(x)$ restricted to vertical strip is completely contained in intersection.



Formal Arguement :

Let $\varepsilon = \frac{1}{2}$ and $\delta > 0$

<u>Claim</u> $\quad 0 < |x-0| < \delta \not\Rightarrow |f(x) - L| < \frac{1}{2}$

<u>Proof</u>

Assume $\quad 0 < |x-0| < \delta \Rightarrow |f(x) - L| < \frac{1}{2}$

Let x_1 be rational such that $0 < |x_1 - 0| < \delta$

and x_2 be irrational such that $0 < |x_2 - 0| < \delta$

$\Rightarrow \quad |f(x_1) - L| < \frac{1}{2}$ and $|f(x_2) - L| < \frac{1}{2}$

$\Rightarrow \quad |-L| = |L| < \frac{1}{2}$ and $|1 - L| < \frac{1}{2}$

Triangle inequality

$1 = (1-L) + -L \Rightarrow 1 < |1-L| + |L| < \frac{1}{2} + \frac{1}{2} = 1$

<u>Contradiction</u> Hence no such $\delta > 0$ exists.

§2.5

Q4/ $[-3,-2)$, $(-2,-1)$, $(-1,0]$, $(0,1)$, $(1,3]$

Q12/ $\lim_{x \to 2} 2x+1 = 5 \neq 0$

$\lim_{x \to 2} x^2+5x = 14$ (Polynomial continuous on $\mathbb{R}$)

Quotient Law $\Rightarrow \lim_{x \to 2} \frac{x^2+5x}{2x+1} = \frac{14}{5} = \frac{2^2+5.2}{2.2+1} = g(2)$

$\Rightarrow g(x)$ continuous at $x=2$

Q16/ Let a be in $(-\infty,-2)$

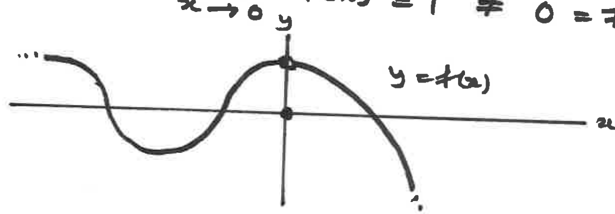
$\lim_{x \to a} 3x+6 = 3a+6$. If $a<-2 \Rightarrow 3a+6 \neq 0$

$\lim_{x \to a} x-1 = a-1$

$\Rightarrow \lim_{x \to a} \frac{x-1}{3x+6} = \frac{a-1}{3a+6} = g(a) \Rightarrow g(x)$ cts on $(-\infty,-2)$.

Q21/ $\lim_{x \to 0^-} \cos(x) = \lim_{x \to 0^+} 1-x^2 = 1 \Rightarrow \lim_{x \to 0} f(x) = 1 \neq 0 = f(0)$

$\Rightarrow f(x)$ discontinuous at $x=0$



Q28/ $-1 \leq \cos(\pi t) \Rightarrow 2 + \cos(\pi t) \neq 0$ for all t

$\lim_{t \to a} (2 + \cos(\pi t)) = 2 + \cos(\pi a)$ (polynomials and cos functions and their compositions are continuous on $\mathbb{R}$)

$\Rightarrow 2 + \cos(\pi t)$ cts on $\mathbb{R}$

e^t and $\sin(t)$ are continuous on $\mathbb{R} \Rightarrow e^{\sin(t)}$ cts on $\mathbb{R}$

$\Rightarrow \dfrac{e^{\sin(t)}}{2 + \cos(\pi t)}$ cts on $\mathbb{R}$.

Q36/ $x + \sin(x)$ cts on $\mathbb{R}$, $\sin(x)$ cts on $\mathbb{R} \Rightarrow \sin(x + \sin(x))$ cts on $\mathbb{R} \Rightarrow \lim_{x \to \pi} \sin(x + \sin(x)) = \sin(\pi + \sin(\pi)) = \sin(\pi) = 0$

Q46/ $\dfrac{x^2 - 4}{x - 2}$ is continuous on $(-\infty, 2)$.

$ax^2 - bx + 3$ continuous on $[2, 3)$

$2x - a + b$ continuous on $[3, \infty)$

$$\lim_{x \to 2^-} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2^-} \frac{(x+2)(x-2)}{x - 2} = \lim_{x \to 2^-} x + 2 = 4$$

$$\lim_{x \to 2^+} ax^2 - bx + 3 = 4a - 2b + 3 = f(2)$$

$$\lim_{x \to 3^-} ax^2 - bx + 3 = 9a - 3b + 3$$

$$\lim_{x \to 3^+} 2x - a + b = 6 - a + b = f(3)$$

Need

$$\begin{aligned} 4a - 2b + 3 &= 4 \\ 9a - 3b + 3 &= 6 - a + b \end{aligned} \Rightarrow \begin{aligned} 4a - 2b &= 1 \\ 10a - 4b &= 3 \end{aligned} \Rightarrow \begin{aligned} 8a - 4b &= 2 \\ 10a - 4b &= 3 \end{aligned}$$

$$\Rightarrow 2a = 1 \Rightarrow a = \frac{1}{2} \Rightarrow 2 - 2b = 1 \Rightarrow b = \frac{1}{2}$$

Hence if $a = b = \frac{1}{2}$ $f(x)$ is cts on $\mathbb{R}$.

Q4f

$$\lim_{x \to 2} \left(3f(x) + f(x)g(x) \right)$$

$$= 3 \lim_{x \to 2} f(x) + \lim_{x \to 2} f(x) \lim_{x \to 2} g(x) = 36$$

$$\Rightarrow 3f(2) + f(2)g(2) = 36$$

$$\Rightarrow f(2) = \frac{36}{3 + g(2)} = \frac{36}{3 + 6} = 4.$$