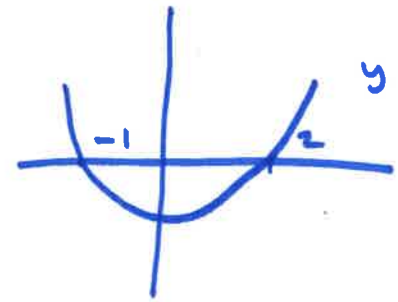
HW 10 Solutions

§4.7

Q5 Vertical Distance at x in $[-1,2]$ $= |x^2 - x - 2|$

$x^2 - x - 2 = (x-2)(x+1) \Rightarrow$

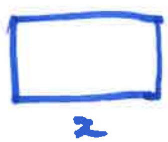


$\Rightarrow |x^2 - x - 2| = 2 + x - x^2$ on $[-1,2]$

Max value is at midpoint of $[-1,2]$, ie $x = \frac{1}{2}$

$\Rightarrow$ Max vertical distance is $2 + \frac{1}{2} - \frac{1}{4} = \frac{9}{4}$

Q7



$Q = xy$

Constraint : $2x + 2y = 100$ $\quad x, y \geq 0$

$\Rightarrow y = 50 - x \Rightarrow Q = x(50 - x) = f(x)$ $\quad x$ in $[0, 50]$

$f'(x) = 50 - 2x = 0 \Rightarrow x = 25$

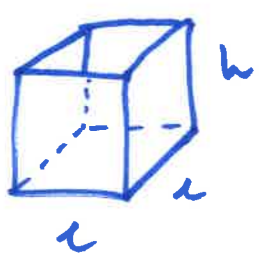
$\Rightarrow x = 25$ gives absolute max $\Rightarrow$ $x = 25$ ft, $y = 25$ ft gives max area

Q9/ $Y = f(N) = \dfrac{kN}{1+N^2} \Rightarrow f'(N) = \dfrac{k(1+N^2) - 2N(kN)}{(1+N^2)^2}$

$= \dfrac{k - kN^2}{(1+N^2)^2} \Rightarrow f'(N) = 0 \Leftrightarrow N = 1 \quad (N \geq 0)$

$f'(N) \Rightarrow N = 1$ givs best yield

Q15/



$Q = h\ell^2$

Contraint : $\ell^2 + 4\ell h = 1200 \quad (\ell, h \geq 0)$

$\Rightarrow h = \dfrac{1200 - \ell^2}{4\ell}$

$\Rightarrow Q = \dfrac{1200 - \ell^2}{4\ell} \cdot \ell^2 = \dfrac{(1200 - \ell^2)\ell}{4} = f(\ell)$

$\Rightarrow f(\ell) = 300\ell - \frac{1}{4}\ell^3 \Rightarrow f'(\ell) = 300 - \frac{3}{4}\ell^2$

$\Rightarrow f'(\ell) = 0 \Rightarrow \ell^2 = 400 \Rightarrow \ell = 20 \Rightarrow h = 10$

$+ \quad\quad -$ $\quad f'(x)$

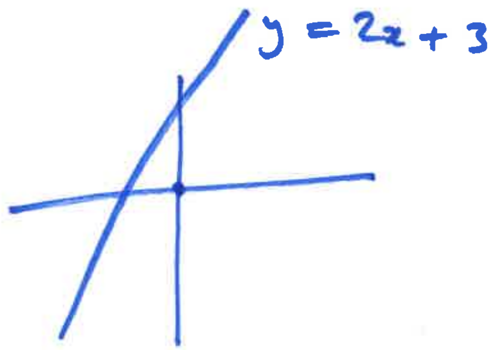
$0 \quad 20$

↖ absolute max

$\Rightarrow$ Max volume $= 20^2 \cdot 10$

$= 4000 \text{ cm}^3$

Q21/



Distance from (x,y) to $(0,0)$

$= \sqrt{x^2+y^2}$

Constraint : $y = 2x+3$

$f(x) = \sqrt{x^2+(2x+3)^2} \Rightarrow f'(x) = \dfrac{2x + 4(2x+3)}{2\sqrt{x^2+(2x+3)^2}}$

$f'(x) = 0 \iff 10x + 12 = 0 \iff x = \dfrac{-12}{10} = \dfrac{-6}{5}$

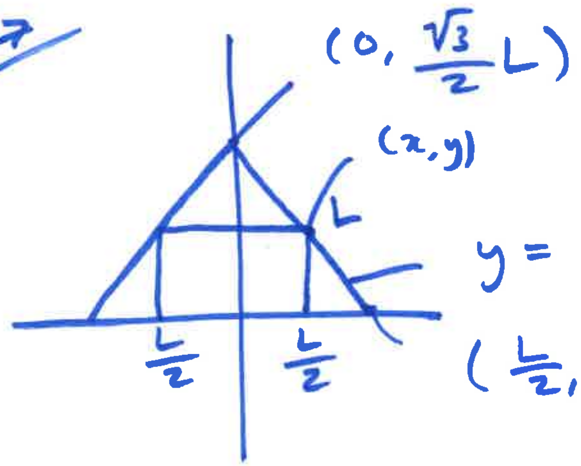
$- \quad\quad +$ $\quad f'(x)$

$\frac{-6}{5}$

↖ absolute min

$\Rightarrow \left(\frac{-6}{5}, 2\left(\frac{-6}{5}\right)+3\right)$ is closest point to $(0,0)$.

Q27



$y = -\sqrt{3}\,x + \frac{\sqrt{3}}{2}L$

$(\frac{L}{2}, 0)$

$Q = 2xy$

Constraint : $y = -\sqrt{3}\,x + \frac{\sqrt{3}}{2}L$

$\Rightarrow Q = 2x(-\sqrt{3}\,x + \frac{\sqrt{3}}{2}L) = f(x)$

$(0 \le x \le \frac{L}{2})$

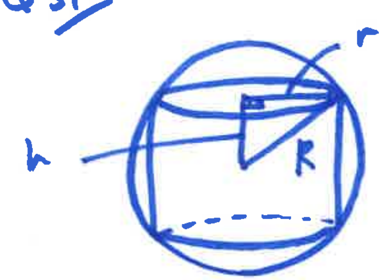
$\Rightarrow f'(x) = -4\sqrt{3}\,x + \sqrt{3}\,L$

$f'(x) = 0 \iff x = \frac{L}{4}$

$f'(x)$: $+$ on $(0, \frac{L}{4})$, $-$ on $(\frac{L}{4}, \frac{L}{2})$

$\Rightarrow$ Area is largest when base is $\frac{L}{2}$ and height is $\frac{\sqrt{3}}{4}L$

Q31



$Q = 2h \cdot \pi r^2 \qquad 0 \le h \le R$

Constraint : $r^2 + h^2 = R^2 \Rightarrow r^2 = R^2 - h^2$

$\Rightarrow Q = f(h) = 2\pi h(R^2 - h^2)$

$\Rightarrow f'(h) = 2\pi R^2 - 6\pi h^2$

$$f'(h) = 0 \iff h^2 = \frac{1}{3}R^2 \Rightarrow h = \frac{R}{\sqrt{3}}$$

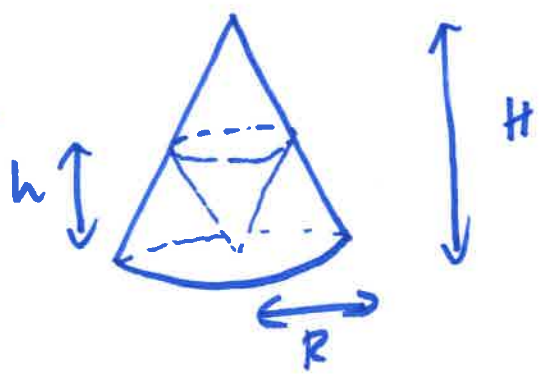
$$\begin{array}{c|ccc} & 0 & \frac{R}{\sqrt{3}} & R \\ \hline f'(h) & + & & - \end{array}$$

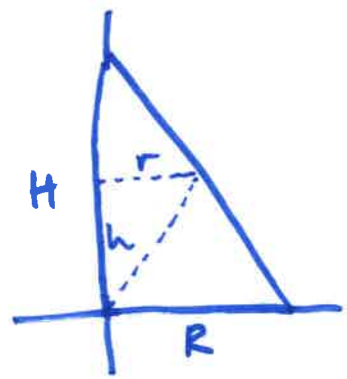
$\Rightarrow$ Volume is max when $h = \frac{R}{\sqrt{3}}$ $\left(\Rightarrow r = \sqrt{R^2 - \frac{R^2}{3}} = \frac{\sqrt{2}R}{\sqrt{3}} \right)$

$\Rightarrow$ Max volume is $2 \cdot \frac{R}{\sqrt{3}} \cdot \pi \cdot \frac{2}{3}R^2$

$$= \frac{4}{3\sqrt{3}}\pi R^3.$$

Q43





Q = Volume of cone with base radius r and height h

$$= \frac{1}{3}\pi r^2 h$$

Constraint :

$$\frac{H}{R} = \frac{H-h}{r} \quad \left(\text{Similar triangles}\right)$$

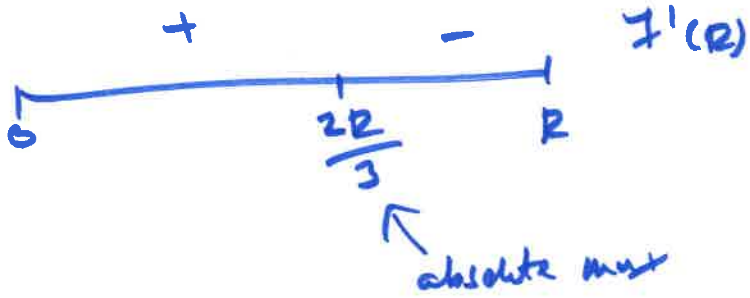
$\Rightarrow H - h = \frac{H}{R}r \Rightarrow h = H - \frac{H}{R}r$

$\Rightarrow Q = f(r) = \frac{1}{3}\pi r^2 \left(H - \frac{H}{R}r \right) = \frac{H}{3}\pi r^2 - \frac{H}{3R}\pi r^3$

$\Rightarrow f'(r) = \frac{2H}{3}\pi r - \frac{H}{R}\pi r^2$

$f'(r) = 0 \iff r = 0$ or $\frac{2H}{3}\pi - \frac{H}{R}\pi r = 0$

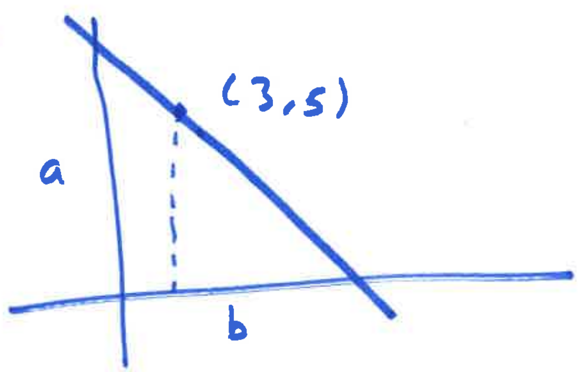
$\Rightarrow \quad r = \frac{2R}{3}$



$\Rightarrow$ Max volume is when

$r = \frac{2R}{3} \Rightarrow h = \frac{H}{3}$.

Q54



$Q = \frac{1}{2} ab \qquad a > 5, \quad b > 3$

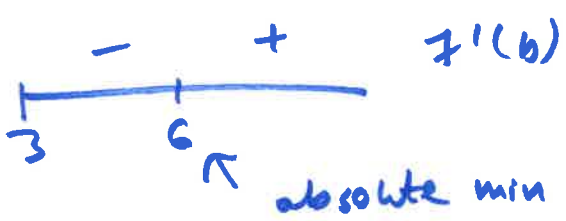
Contraint : $\frac{b-3}{5} = \frac{b}{a}$ (Similar triangles)

$\Rightarrow \quad a = \frac{5b}{b-3}$

$\Rightarrow \quad Q = f(b) = \frac{1}{2} \cdot \frac{5b^2}{b-3}$

$f'(b) = \frac{1}{2} \frac{10b(b-3) - 5b^2}{(b-3)^2} = \frac{1}{2} \cdot \frac{5b^2 - 30b}{(b-3)^2}$

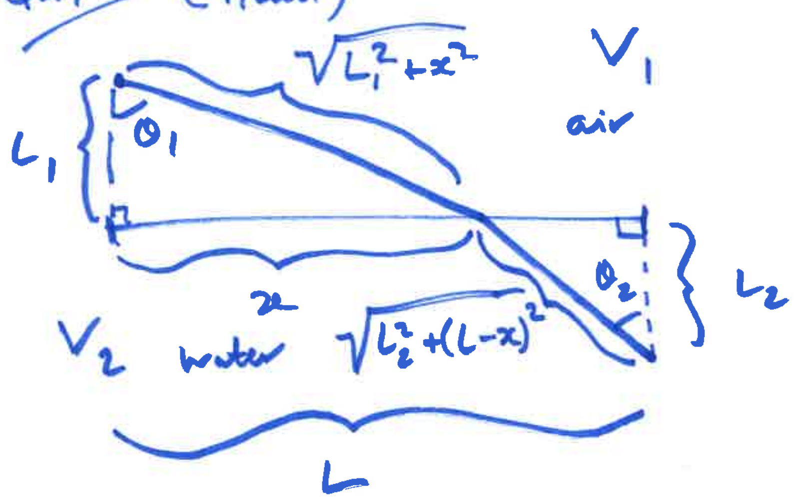
$$f'(b) = 0 \iff 5b^2 - 30b = 0 \iff \cancel{b = 0} \text{ or } b = 6$$



$\Rightarrow$ $b = 6$ and $a = 10$ gives min area

$\Rightarrow$ $y = \frac{-10}{6}x + 10$ is equation of the line cutting off minimal area.

Q71 (Hard)



time in air

time in water

$$t_1 = \frac{\sqrt{L_1^2 + x^2}}{V_1}, \quad t_2 = \frac{\sqrt{L_2^2 + (L-x)^2}}{V_2}$$

Need to minimize

$$f(x) = t_1 + t_2 = \frac{\sqrt{L_1^2 + x^2}}{V_1} + \frac{\sqrt{L_2^2 + (L-x)^2}}{V_2}$$

$$T'(x) = \frac{2x}{2V_1\sqrt{L_1^2+x^2}} + \frac{-2(L-x)}{2V_2\sqrt{L_2^2+(L-x)^2}} = \frac{\frac{x}{\sqrt{L_1^2+x^2}}}{V_1} - \frac{\frac{L-x}{\sqrt{L_2^2+(L-x)^2}}}{V_2}$$

$$= \frac{\sin(\theta_1(x))}{V_1} - \frac{\sin(\theta_2(x))}{V_2} \qquad \left(\begin{array}{l}\theta_1(x) \text{ and } \theta_2(x) \\ \text{are both functions in } x\end{array}\right)$$

$$T'(x) = 0 \iff \frac{\sin(\theta_1(x))}{\sin(\theta_2(x))} = \frac{V_1}{V_2}$$

Claim 1 There is a value of x for which this happens

Proof $\theta_1(x)$ and $\theta_2(x)$ are both continuous functions in x

$\Rightarrow$ $T'(x)$ is continuous.

$$T'(0) = \frac{\sin(0)}{V_1} - \frac{\sin(\theta_2(0))}{V_2}, \quad \frac{\pi}{2} > \theta_2(0) > 0$$

$\Rightarrow$ $T'(0) < 0$

$$T'(L) = \frac{\sin(\theta_1(L))}{v_1} - \frac{\sin(0)}{v_2} = \frac{\sin(\theta_1(L))}{v_1}$$

$\frac{\pi}{2} > \theta_1(L) > 0 \implies T'(L) > 0$

$\implies$ By I.V.T there exists x_0 in $[0, L]$ s.t.

$$T'(x_0) = 0 \quad \text{, ie when} \quad \frac{\sin(\theta_1(x_0))}{\sin(\theta_2(x_0))} = \frac{v_1}{v_2}$$

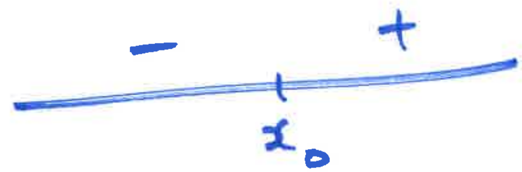
<u>Claim 2</u> This gives an absolute min for $T(x)$.

<u>Proof</u> $\theta_1(x)$ is <u>increasing</u> in x and has range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\theta_2(x)$ is <u>decreasing</u> in x and has range $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\implies \sin(\theta_1(x))$ increasing and $\sin(\theta_2(x))$ decreasing.

$$\implies T'(x) = \frac{\sin(\theta_1(x))}{v_1} - \frac{\sin(\theta_2(x))}{v_2} \quad \text{is increasing}$$

$\Rightarrow$ 

$T'(x)$ $\Rightarrow$ when $\frac{\sin(\theta_1)}{\sin(\theta_2)} = \frac{v_1}{v_2}$

the time taken is minimized.

§4.4

Q4 $x^6 - \frac{8}{5}x^5 - 3x^3 + C$

Q7 $\frac{7}{2/5+1} x^{(2/5+1)} + \frac{8}{-\frac{4}{5}+1} x^{(-4/5+1)} + C$

Q10 $e^2 x + C$

Q15 $g(t) = t^{-\frac{1}{2}} + t^{1/2} + t^{3/2}$

$G(t) = \frac{1}{-\frac{1}{2}+1} t^{-\frac{1}{2}+1} + \frac{1}{1/2+1} t^{1/2+1} + \frac{1}{3/2+1} t^{(3/2+1)} + C$

Q21 $f(x) = 2x + 4 - x^{-2}$

$F(x) = x^2 + 4x + \frac{1}{x} + C$

Q37 $f'(t) = \sec^2(t) + \sec(t)\tan(t)$

$\Rightarrow f(t) = \tan(t) + \sec(t) + C$

$f(\pi/4) = 1 + \sqrt{2} + C = -1 \Rightarrow C = -2 - \sqrt{2}$

$\Rightarrow \quad f(t) = \tan(t) + \sec(t) - 2 - \sqrt{2}$

Q 41

$f''(\theta) = \sin\theta + \cos\theta \Rightarrow f'(\theta) = -\cos\theta + \sin\theta + C$

$f'(0) = 4 \Rightarrow C = 5 \Rightarrow f'(\theta) = -\cos\theta + \sin\theta + 5$

$\Rightarrow f(\theta) = -\sin\theta - \cos\theta + 5\theta + C$

$f(0) = 3 \Rightarrow C = 4 \Rightarrow f(\theta) = -\sin\theta - \cos\theta + 5\theta + 4$

Q48

$f'''(x) = \cos(x) \Rightarrow f''(x) = \sin(x) + C$

$f''(0) = 3 \Rightarrow C = 3 \Rightarrow f''(x) = \sin(x) + 3$

$\Rightarrow f'(x) = -\cos(x) + 3x + C$

$f'(0) = 2 \Rightarrow C = 3 \Rightarrow f'(x) = -\cos(x) + 3x + 3$

$\Rightarrow f(x) = -\sin(x) + \frac{3}{2}x^2 + 3x + C$

$f(0) = 1 \Rightarrow C = 1 \Rightarrow f(x) = -\sin(x) + \frac{3}{2}x^2 + 3x + 1.$

Q40

$f'(x) = 3 - 4x \Rightarrow f(x) = 3x - 2x^2 + C$

$f(2) = 5 \Rightarrow 5 = 6 - 8 + C \Rightarrow C = 7$

$\Rightarrow f(x) = 3x - 2x^2 + 7 \Rightarrow f(1) = 8$

Q61

$a(t) = 2t + 1 \Rightarrow v(t) = t^2 + t + C$

$v(0) = -2 \Rightarrow v(t) = t^2 + t - 2$

$\Rightarrow s(t) = \frac{1}{3}t^3 + \frac{1}{2}t^2 - 2t + C$

$s(0) = 3 \Rightarrow s(t) = \frac{1}{3}t^3 + \frac{1}{2}t^2 - 2t + 3$

Q63

$a(t) = 10\sin(t) + 3\cos(t) \Rightarrow v(t) = -10\cos(t) + 3\sin(t) + C$

$\Rightarrow s(t) = -10\sin(t) - 3\cos(t) + Ct + D$

$s(0) = 0 \Rightarrow -3 + D = 0 \Rightarrow D = 3$

$s(2\pi) = 12 \Rightarrow -3 + 2\pi C + D = 12 \Rightarrow C = \frac{12}{2\pi} = \frac{6}{\pi}$

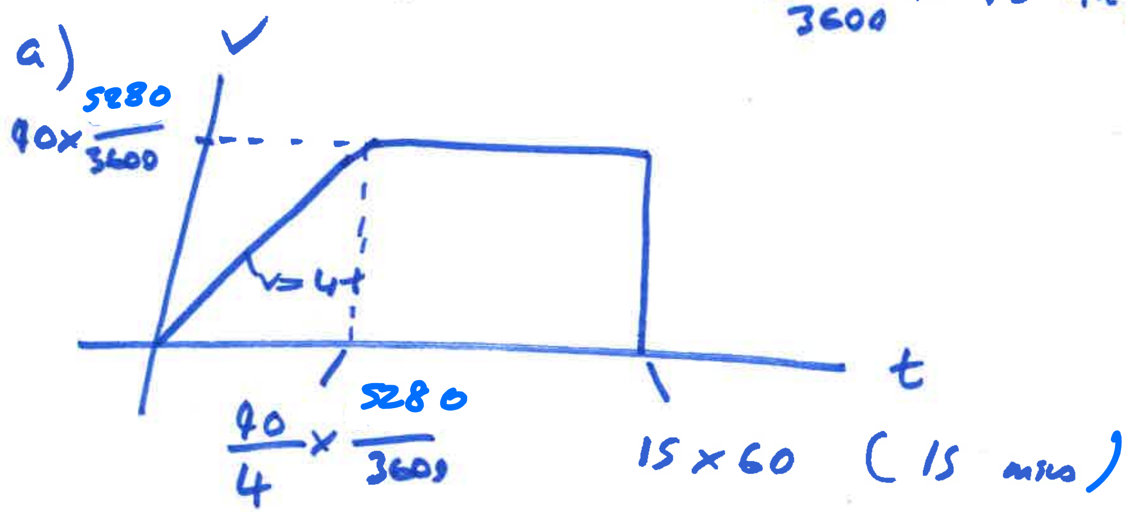
$\Rightarrow \quad s(t) = -10\sin(t) - 3\cos(t) + \frac{6}{\pi}t + 3$

Q66/ $a(t) = a \Rightarrow v(t) = at + v_0 \Rightarrow s(t) = \frac{1}{2}at^2 + v_0 t + s_0$

Q74/ This question is awful because we have to switch between units. 1 mile per hour $= \frac{5280}{3600}$ ft/sec

$\Rightarrow$ 90 miles per hour $= \frac{5280}{3600} \times 90$ ft/s

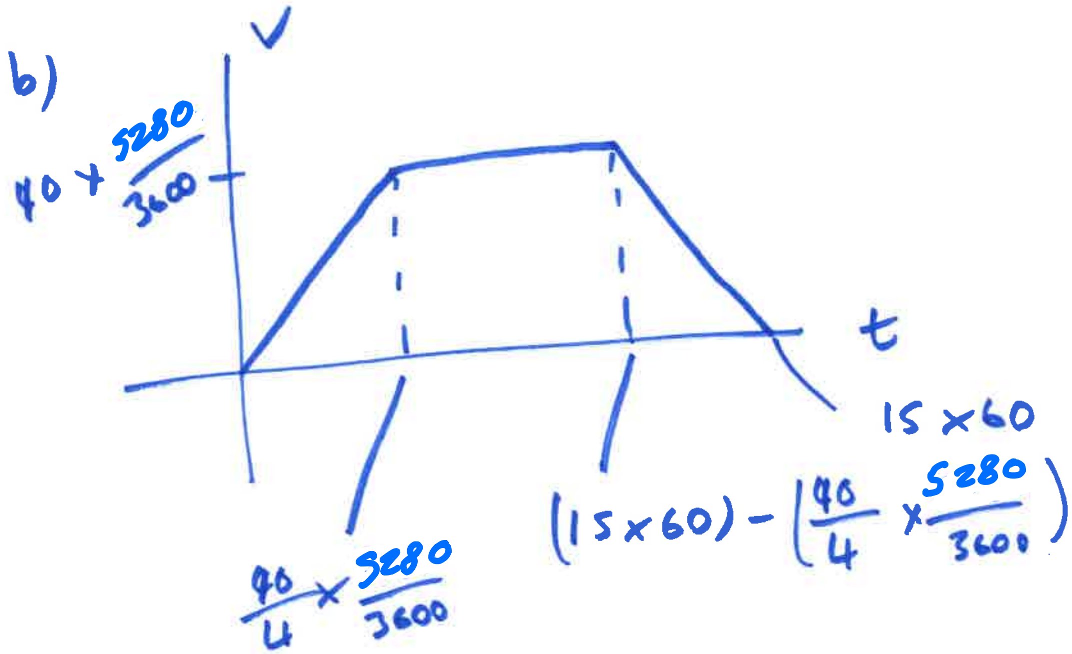
a)



Max distance travelled in 15 mins $= \left(\frac{90}{4} \times \frac{5280}{3600}\right) \times \left(90 \times \frac{5280}{3600}\right) \times \frac{1}{2}$

$+ \; 90 \times \frac{5280}{3600} \times \left(15 \times 60 - \frac{90}{4} \times \frac{5280}{3600}\right)$ ft.

b)



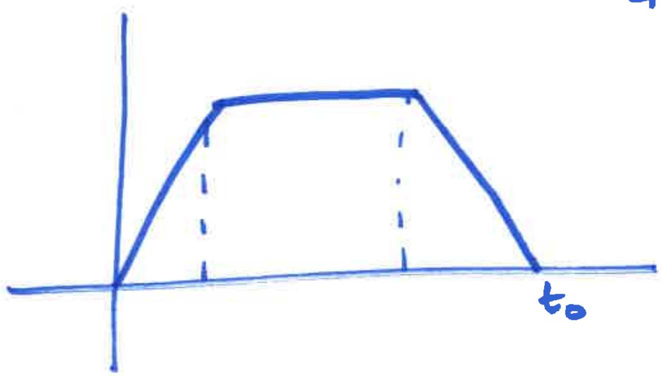
Max distance

$= $ area under graph

$= 40 \times \frac{5280}{3600} \times \left((15\times60) - \frac{90}{4} \times \frac{5280}{3600}\right)$

ft.

c)



$45 = \text{Distance} = \left(t_0 - \frac{90}{4} \cdot \frac{5280}{3600}\right) \cdot \left(40 \times \frac{5280}{3600}\right)$

$\Rightarrow \quad t_0 = \dfrac{45}{40 \times \frac{5280}{3600}} + \frac{90}{4} \cdot \frac{5280}{3600}$ sec

d) Distance $= \left((37.5 \times 60) - \frac{90}{4} \cdot \frac{5280}{3600}\right) \cdot \left(40 \times \frac{5280}{3600}\right)$ ft.