Volumes

Q : What is the volume of a sphere of radius 1 ?

First solved by

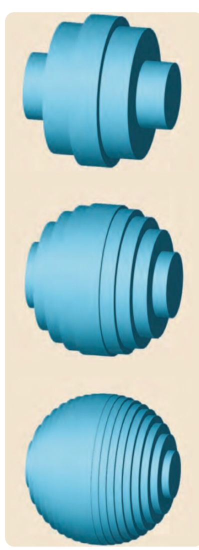
Archimedes

(287 BC – 212 BC)

We know volume of a cylinder

Basic Strategy : Approximate sphere by cylinders to higher and higher accuracy.

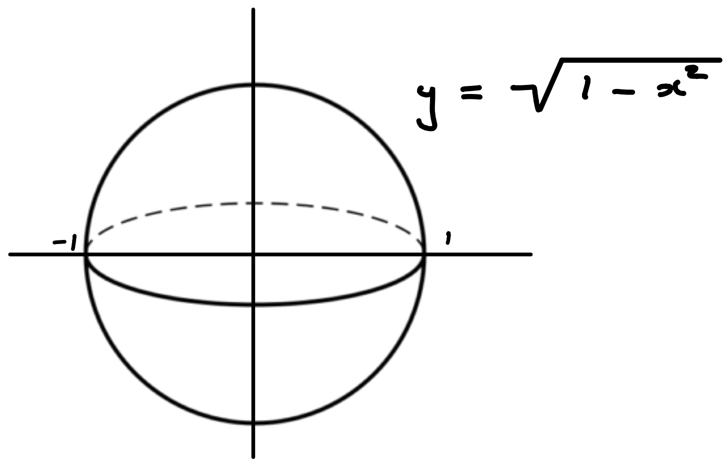
Picture :



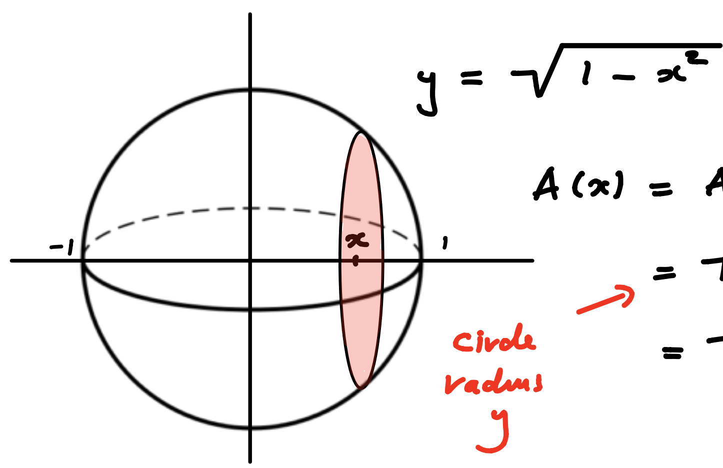
More accurate approximations

Step 1 : Center sphere at $(0,0)$ in xy-plane

Observe top is given by $y = \sqrt{1-x^2}$



Step 2 : Define $A(x)$ = area of cross-section of sphere at x



$$A(x) = \text{Area}(\ \ ) = \pi y^2 = \pi(1-x^2)$$

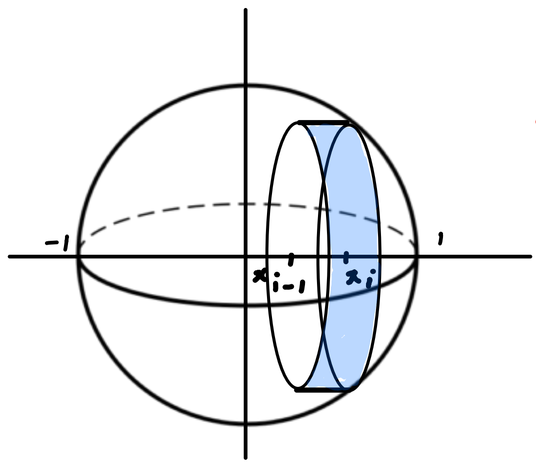
Step 3 : For each n, a natural number, subdivide $[-1,1]$ into n equal subintervals of length $\Delta x = \frac{2}{n}$.

$\frac{1-(-1)}{n}$

<u>Step 4</u> : Over the subinverval $[x_{i-1}, x_i]$ draw the cylinder of height Δx and radius $f(x_i)$

Volume of i^{th} cylinder $= \underbrace{\pi f(x_i)^2}_{A(x_i)} \Delta x$

Area of base times height



<u>Step 5</u> Sum the volumes cylinders and take a limit as $n \to \infty$

Riemann Sum

$$\Rightarrow \text{Volume of Sphere} = \lim_{n \to \infty} \sum_{i=1}^{n} A(x_i)\Delta x$$

By definition of definite integral

$$= \int_{-1}^{1} A(x)\,dx$$

$$= \int_{-1}^{1} \pi(1-x^2)\,dx$$

FTC (Archimedes didn't know this)

$$= \pi x - \frac{\pi}{3}x^3 \Big|_{-1}^{1} = \frac{4}{3}\pi$$

More generally :

Volume of a sphere of radius r

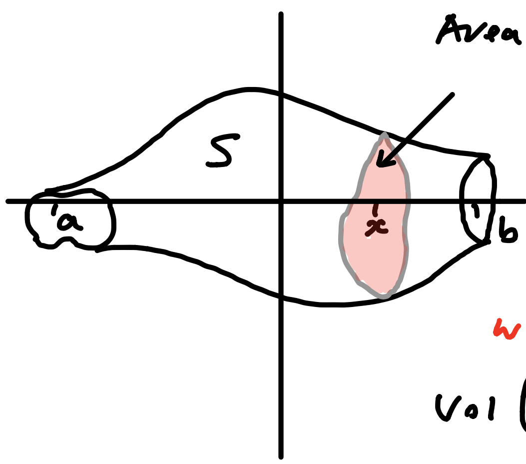
$$= \int_{-r}^{r} \overbrace{\pi \left(r^2 - x^2 \right)}^{A(x)} \, dx$$

$$= \pi r^2 x - \frac{\pi}{3} x^3 \Big|_{-r}^{r} = \frac{4}{3} \pi r^3$$

Awesome

What about different solid shapes?

Same logic $\Rightarrow$



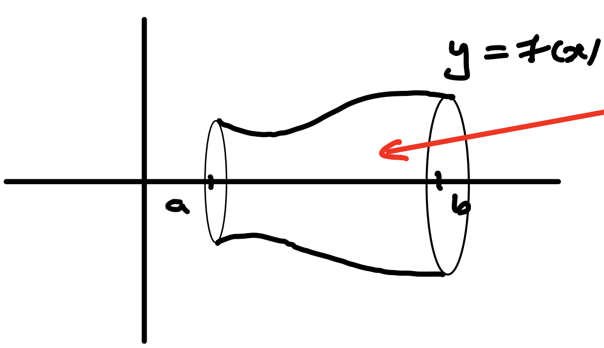
Area( ) $= A(x)$

$$\Rightarrow \mathrm{Vol}(S) = \int_a^b A(x) \, dx$$

works because

Vol( ) = Area of base times height

Important Example : Solid of revolution



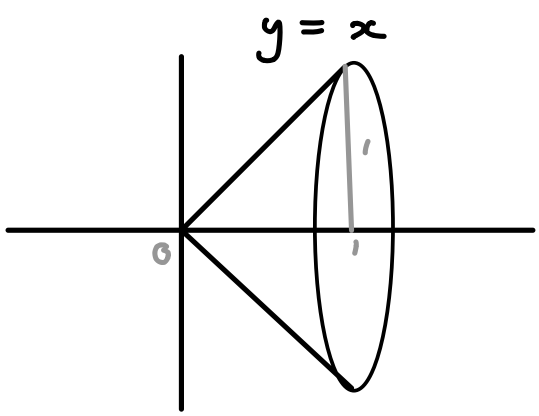
Solid formed by rotating $y = f(x)$ around x-axis.

$\Rightarrow$

$$\boxed{\text{Volume of Solid of revolution of } y=f(x) \text{ between } x=a \text{ and } x=b \;=\; \int_a^b \underbrace{\pi (f(x))^2}_{A(x)} \, dx}$$

Example

1/ What is the volume of a cone with radius 1 and height 1 ?

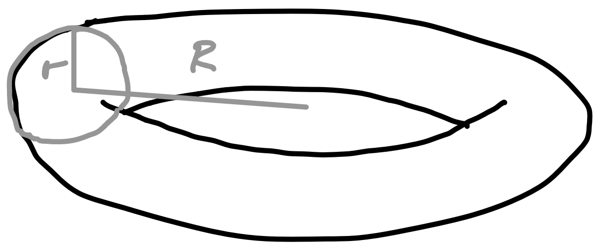


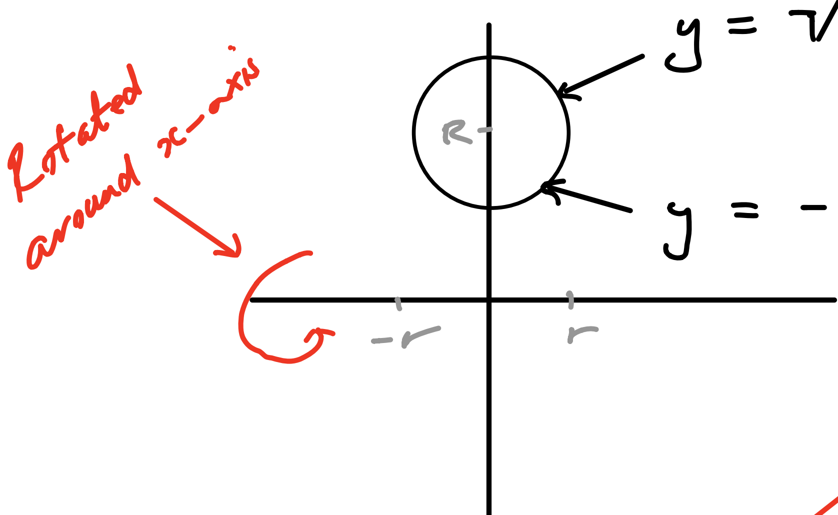
$$\Rightarrow \quad \text{Volume} = \int_0^1 \pi x^2 \, dx = \frac{\pi}{3} x^3 \Big|_0^1 = \frac{\pi}{3}$$

More generally :

$$\text{Volume of cone of radius } r \text{ and height } h = \int_0^h \pi \left(\frac{r}{h} x\right)^2 dx = \frac{1}{3}\pi r^2 h$$

2/ What is the volume of a solid circular donut with the following dimensions:





$y = \sqrt{r^2 - x^2} + R$

$y = -\sqrt{r^2 - x^2} + R$

area of large circle

$$\Rightarrow A(x) = \pi\left(\sqrt{r^2 - x^2} + R\right)^2 - \pi\left(-\sqrt{r^2 - x^2} + R\right)^2$$

area of small circle

$$= 4R\sqrt{r^2 - x^2}$$

$$\Rightarrow \text{Volume} = \int_{-r}^{r} \pi \cdot 4R\sqrt{r^2 - x^2}\,dx = 4\pi R \overbrace{\int_{-r}^{r} \sqrt{r^2 - x^2}\,dx}^{\text{area of semi-circle radius } r}$$

$$= 4\pi R \cdot \tfrac{1}{2}\pi r^2 = 2\pi R \cdot \pi r^2$$

Conclusion : To calculate volume of solid S :

1/ Sit S in xy-plane with $x=a$ and $x=b$ at ends

2/ Determine cross-section area function $A(x)$

3/ $\mathrm{Vol}(S) = \int_a^b A(x)\,dx$ ← Calculate using FTOC.