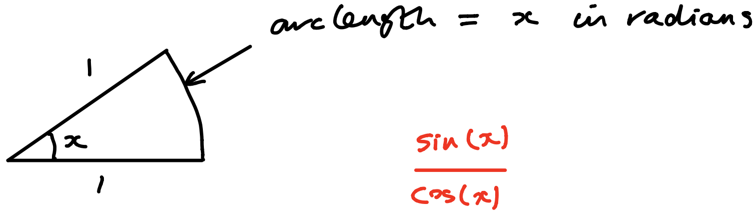
Trigonometric Derivatives

Recall :



$f(x) = \sin(x) \,/\, \cos(x) \,/\, \tan(x)$, x is always in radians

$\tan(x) = \dfrac{\sin(x)}{\cos(x)}$

Q/ $\dfrac{d}{dx}(\sin(x)) = ?$

$$\frac{d}{dx}(\sin(x)) = \lim_{h \to 0} \frac{\sin(x+h) - \sin(x)}{h}$$

(Trig. Identity) $$= \lim_{h \to 0} \frac{\sin(x)\cos(h) + \sin(h)\cos(x) - \sin(x)}{h}$$

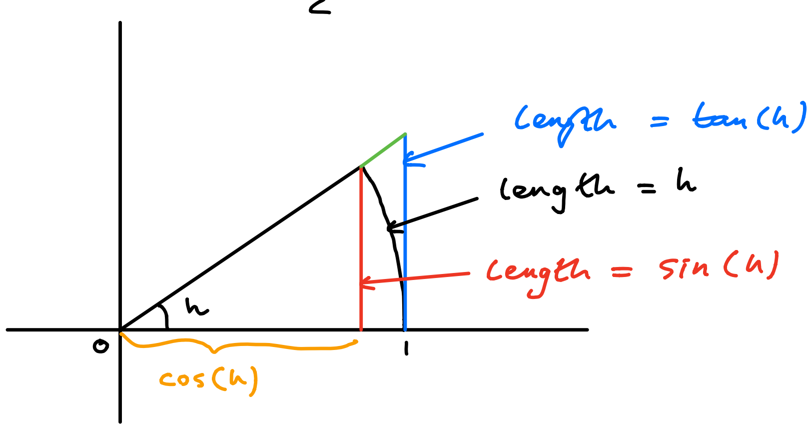
$$= \lim_{h \to 0} \left(\frac{\sin(h)}{h} \cos(x) - \sin(x) \frac{1 - \cos(h)}{h} \right)$$

Limit Laws

$$= \left(\lim_{h \to 0} \frac{\sin(h)}{h} \right) \cdot \cos(x) - \sin(x) \left(\lim_{h \to 0} \frac{1 - \cos(h)}{h} \right)$$

Q/ $\displaystyle \lim_{h \to 0} \frac{\sin(h)}{h} = ?$, $\displaystyle \lim_{h \to 0} \frac{1 - \cos(h)}{h} = ?$

Let $0 < h < \frac{\pi}{2}$



$$\Rightarrow \quad \sin(h) \le h \le \tan(h)$$

$$\Rightarrow \quad 1 \le \frac{h}{\sin(h)} \le \frac{\tan(h)}{\sin(h)} = \frac{1}{\cos(h)}$$

$$\lim_{h \to 0} \cos(h) = 1 \Rightarrow \lim_{h \to 0} \frac{1}{\cos(h)} = \frac{1}{1} = 1$$

Squeeze Theorem $\Rightarrow$

$$\lim_{h \to 0^+} \frac{h}{\sin(h)} = 1 \Rightarrow \lim_{h \to 0^+} \frac{\sin(h)}{h} = 1 \Rightarrow \boxed{\lim_{h \to 0} \frac{\sin(h)}{h} = 1}$$

<u>Observation</u>

$$\lim_{h \to 0} \frac{1 - \cos(h)}{h} = \lim_{h \to 0} \frac{1 - \cos(h)}{h} \cdot \frac{1 + \cos(h)}{1 + \cos(h)}$$

$$= \lim_{h \to 0} \frac{1 - \cos^2(h)}{h(1 + \cos(h))} = \lim_{h \to 0} \frac{\sin^2(h)}{h(1 + \cos(h))}$$

$$= \lim_{h \to 0} \left(\frac{\sin(h)}{h}\right)^2 \cdot \frac{h}{1 + \cos(h)}$$

$$= \left(\lim_{h \to 0} \frac{\sin(h)}{h}\right)^2 \cdot \frac{\lim_{h \to 0} h}{\lim_{h \to 0} 1 + \cos(h)} = 1^2 \cdot \frac{0}{2} = 0$$

$$\Rightarrow \quad \lim_{h \to 0} \frac{1 - \cos(h)}{h} = 0$$

<u>Conclusion</u>

$$\boxed{\frac{d}{dx} \sin(x) = \cos(x)}$$

Same Method $\Rightarrow$ $\boxed{\frac{d}{dx}\cos(x) = -\sin(x)}$

Good Exercise

Other Trigonometric Functions : $\tan(x) = \frac{\sin(x)}{\cos(x)}$,

$\sec(x) = \frac{1}{\cos(x)}$, $\csc(x) = \frac{1}{\sin(x)}$, $\cot(x) = \frac{1}{\tan(x)} = \frac{\cos(x)}{\sin(x)}$

secant x cosecant x cotangent x

Quotient Rule $\Rightarrow$

$$\frac{d}{dx}\tan(x) = \frac{\cos^2(x)+\sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x)$$

$$\frac{d}{dx}\sec(x) = \frac{\sin(x)}{\cos^2(x)} = \tan(x)\sec(x)$$

$$\frac{d}{dx}\csc(x) = \frac{-\cos(x)}{\sin^2(x)} = -\cot(x)\csc(x)$$

$$\frac{d}{dx}\cot(x) = \frac{-\sin^2(x)-\cos^2(x)}{\sin^2(x)} = \frac{-1}{\sin^2(x)} = -\csc^2(x)$$

Remark Don't waste your time memorizing these. Just focus on $\frac{d}{dx}(\sin(x))$, $\frac{d}{dx}(\cos(x))$ and the definitions.

Example 1, $\frac{d^2}{dx^2}\tan(x) = ?$

$$\frac{d}{dx}(\tan(x)) = \sec^2(x)$$

$$\Rightarrow \frac{d^2}{dx^2}(\tan(x)) = \frac{d}{dx}\sec(x)\sec(x) = \left(\frac{d}{dx}(\sec(x))\right)\sec(x) + \sec(x)\left(\frac{d}{dx}(\sec(x))\right)$$

$$= \tan(x)\sec(x) \cdot \sec(x) + \sec(x)\tan(x)\sec(x)$$

$$= 2\tan(x)\sec^2(x) = \frac{2\sin(x)}{\cos^3(x)}$$

2/ $$\frac{d^9}{dx^9}(\sin(x)) = ?$$

$$\frac{d}{dx}(\sin(x)) = \cos(x)$$

$$\frac{d^2}{dx^2}(\sin(x)) = -\sin(x)$$

$$\frac{d^3}{dx^3}(\sin(x)) = -\cos(x)$$

$$\frac{d^4}{dx^4}(\sin(x)) = \sin(x)$$

Repeats in blocks of four

$$\Rightarrow \frac{d^9}{dx^9}(\sin(x)) = \cos(x)$$