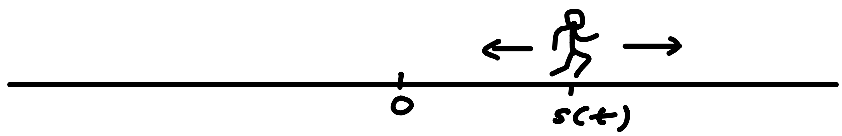
Tangents and Velocity

Motion in a straight line :

t = time in seconds

$s(t)$ = position at time t (in meters from 0)



Q : What is my velocity at a <u>single moment</u> in time ?

Let $a < b$ ← two distinct moments in time

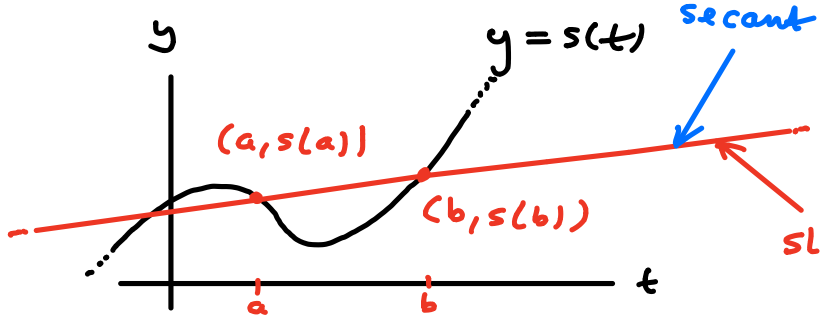
<u>Average</u> velocity between $t=a$ and $t=b$

$$= \frac{\text{Overall displacement between } t=a \text{ and } t=b}{\text{Time elapsed between } t=a \text{ and } t=b}$$

$$= \frac{s(b) - s(a)}{b-a}$$

<u>Observation</u> :

Called secant line



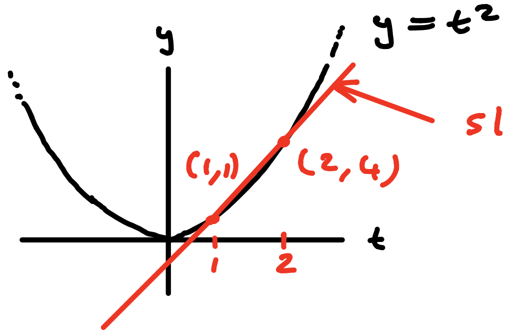
Average velocity between $t=a$ and $t=b$

$$\shortparallel$$

slope $= \dfrac{s(b) - s(a)}{b-a}$

Example

$s(t) = t^2$, $a = 1$, $b = 2$



$$\text{slope} = \frac{4-1}{2-1} = 3 = \text{Average velocity between } t=1 \text{ and } t=2$$

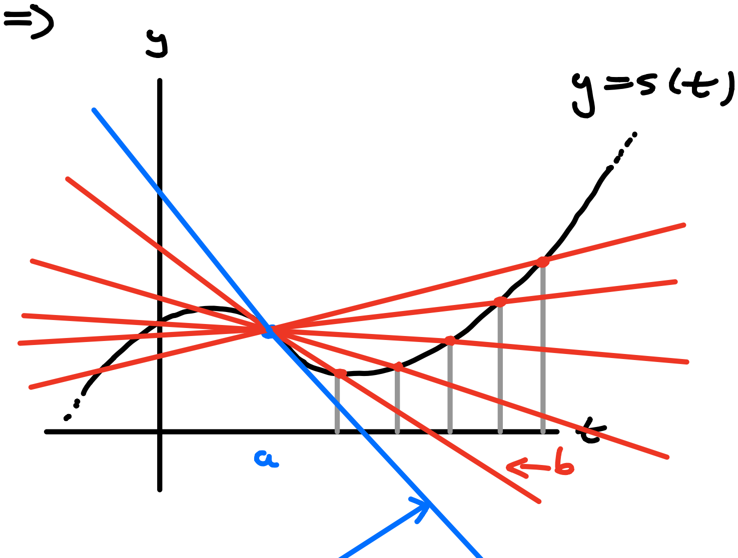
cannot divide by 0

Problem If $a = b$ this is undefined. What does velocity at a single moment even mean?

Strategy : Consider average velocities over smaller and smaller intervals of time.

Key Observation : As b gets closer and closer to a, $(b, s(b))$ gets closer and closer to $(a, s(a))$

$\Rightarrow$



Tangent line at $(a, s(a))$
(The line just kissing graph at $(a, s(a))$)

$\Rightarrow$ As b gets closer and closer to a, the slopes of the secant lines get closer and closer to the slope of the tangent line at $(a, s(a))$

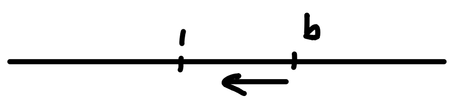
Definition

The *instantaneous* velocity at time $t=a$ is the slope of the tangent line at $(a, s(a))$.

Problem : We only know $(a, s(a))$ is on tangent line. We usually need two points to calculate slope.

Example

$s(t) = t^2$, $a = 1$, $b > 1$



b	Average velocity between $t=1$ and $t=b$
1.1	2.1
1.01	2.01
1.001	2.001
1.0001	2.0001

Looks like it's getting close to 2. Need to be certain.

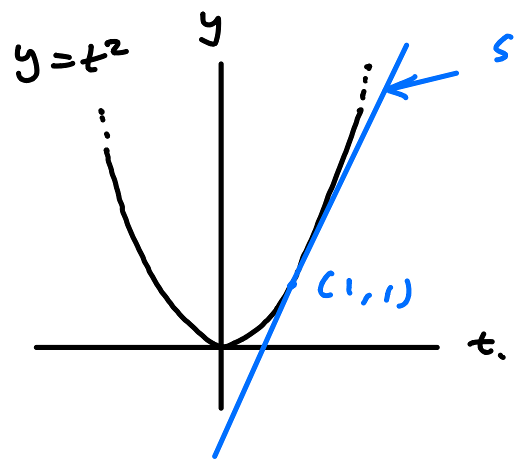
More systematic :

$$\frac{s(b) - s(1)}{b-1} = \frac{b^2 - 1}{b-1}$$

$$b > 1 \Rightarrow \frac{b^2-1}{b-1} = \frac{(b+1)(b-1)}{b-1} = b+1$$

As b gets closer to 1, $b+1$ gets closer to 2

$\Rightarrow$ Slope of tangent line to $y = t^2$ at $(1,1)$ is 2



Slope $= 2 = \dfrac{\text{Instantaneous}}{\text{velocity at } t=1}$

<u>Aim</u> : Develop these ideas in general.