Integration by Substitution

Recall:

Called integrand

Called indefinite integral

$$\int f(x)\,dx = \text{Most general antiderivative of } f(x)$$

On open interval $\longrightarrow$ $= F(x) + C$ ← Constant

Particular antiderivative $(F'(x) = f(x))$

Why it matters:

Definite integral (Net Area)

FTOC : $$\int_a^b f(x)\,dx = F(b) - F(a)$$

Aim : Find antiderivatives for as many functions as possible.

Core Examples/Rules

$$\int x^r\,dx = \frac{1}{r+1}x^{r+1} + C$$ $r \neq -1$

$$\int x^{-1}\,dx = \ln|x| + C$$ On $(0,\infty)$ or $(-\infty,0)$

$$\int b^x\,dx = \frac{1}{\ln(b)}b^x + C$$ $b \neq 1$

$$\int \sin(x)\,dx = -\cos(x) + C \;,\quad \int \cos(x)\,dx = \sin(x) + C$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx \Big/ \int \frac{-1}{\sqrt{1-x^2}}\,dx = \arcsin(x)/\arccos(x) + C$$

$$\int \frac{1}{1+x^2}\,dx = \arctan(x) + C$$

Sum rule in reverse

Constant multiple rule in reverse

$$\int f(x)+g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx \;,\quad \int c f(x)\,dx = c\int f(x)\,dx$$

Integration by Substitution = Chain Rule in reverse.

Chain Rule :

$$\frac{d}{dx} F(g(x)) = F'(g(x)) \cdot g'(x) \overset{F'(x) = f(x)}{=} f(g(x))\, g'(x)$$

$$\Rightarrow \boxed{\int f(g(x))\, g'(x)\, dx = F(g(x)) + C}$$

Examples

1/ $\int \cos(x^2)\, 2x\, dx = ?$

$f(x) = \cos(x)$, $g(x) = x^2 \Rightarrow f(g(x))\, g'(x) = \cos(x^2)\, 2x$

$\int \cos(x)\, dx = \sin(x) + C$, hence let $F(x) = \sin(x)$

$$\Rightarrow \int \underbrace{\cos(x^2) 2x}_{f(g(x))g'(x)}\, dx = \underbrace{\sin(x^2)}_{F(g(x))} + C$$

2/ $\int e^{2\cos(x)} \sin(x)\, dx = ?$

$f(x) = e^x$, $g(x) = \cos(x)$

Careful : $g'(x) = -2\sin(x) \Rightarrow e^{\cos(x)} \sin(x) = \overset{\text{Out by constant}}{\frac{-1}{2}} f(g(x)) g'(x)$

$\int e^x dx = e^x + C$, hence let $F(x) = e^x$

$$\Rightarrow \int e^{\cos(x)} \sin(x)\, dx = \frac{-1}{2} \int f(g(x))\, g'(x)\, dx = \frac{-1}{2} F(g(x)) + C$$

$$= \frac{-1}{2} e^{\cos(x)} + C$$

3/ $\int \frac{\ln(x)}{x}\,dx = ?$

<u>Problem</u> : Not obviously of the form $f(g(x))g'(x)$.

Clever Observation : $\frac{d}{dx}\ln(x) = \frac{1}{x}$ and $\frac{\ln(x)}{x} = \ln(x)\cdot\frac{1}{x}$

Let $f(x) = x$ and $g(x) = \ln(x)$ $\Rightarrow$ $\frac{\ln(x)}{x} = f(g(x))g'(x)$

$\int x\,dx = \frac{1}{2}x^2 + C$, hence let $F(x) = \frac{1}{2}x^2$

$$\Rightarrow \int \frac{\ln(x)}{x}\,dx = \frac{1}{2}(\ln(x))^2 + C$$

<u>Conclusion</u> : For this to work must

A/ spot $f(x)$ and $g(x)$ such that integrand is of form $f(g(x))g'(x)$. ← Could be very hard.

B/ Be able to integrate (find antiderivative) of $f(x)$.

<u>More systematic Approach</u> : Integration by Substitution

1/ Examine Integrand. Is it of the form $f(g(x))\,g'(x)$ up to multiplication by a constant?

<u>Example</u>

$f(x) = e^x$

$g(x) = 2\cos(x)$

$\Rightarrow e^{2\cos(x)}\sin(x) = \frac{-1}{2}f(g(x))g'(x)$

2/ Introduce a new variable u and set $u = g(x)$.

We're going to change the variable to u in the indefinite integral. This is done in 2 stages:

a) Replace dx with expression in du: Just clever notation

$$u = g(x) \Rightarrow \frac{du}{dx} = g'(x) \Rightarrow dx = \frac{du}{g'(x)}$$

Example

$$u = 2\cos(x) \Rightarrow \frac{du}{dx} = -2\sin(x) \Rightarrow dx = \frac{du}{-2\sin(x)}$$

$$\Rightarrow \int e^{2\cos(x)} \sin(x)\, dx = \int e^{2\cos(x)} \cancel{\sin(x)} \frac{du}{-2\cancel{\sin(x)}}$$

$$= \int \frac{-1}{2} e^{2\cos(x)}\, du$$

b) Rewrite integrand in u variable

Example

$u = 2\cos(x)$

$$\int \frac{-1}{2} e^{2\cos(x)}\, du = \int \frac{-1}{2} e^{u}\, du$$

3/ Calculate integral in u-variable.

Example

$$\int \frac{-1}{2} e^{u}\, du = \frac{-1}{2} \int e^{u}\, du = \frac{-1}{2} e^{u} + C$$

4/ Replace u with $g(x)$ to express final answer in x-variable

<u>Example</u>

$$\frac{-1}{2}e^{u} + C = \frac{-1}{2}e^{2\cos(x)} + C$$

<u>Overview of Integration by Substitution</u>

<u>x-world</u> (Original variable)

$$\left(\begin{array}{l} u = g(x) \\ \Rightarrow \dfrac{du}{dx} = g'(x) \\ \Rightarrow dx = \dfrac{du}{g'(x)} \end{array} \right)$$

<u>u-world</u> (New variable)

1/

4/ Replace u with $g(x)$

3/ Calculate integral

$$\int f(g(x))\, g'(x)\, dx = F(g(x)) + C \xleftarrow{} F(u) + C \xleftarrow{} \int f(u)\, du$$

Replace dx

2/ (a)

2/ (b)

Replace $g(x)$ with u

$$\int f(g(x))\, \cancel{g'(x)} \frac{du}{\cancel{g'(x)}} = \int f(g(x))\, du$$

<u>Remark</u> : The strength of this approach is that we only really need to spot $g(x)$ to impliment it.

Examples

1/ $\int \tan(x)\,dx = ?$

$\frac{d}{dx}(\cos(x)) = -\sin(x)$ 1/

$\tan(x) = \frac{\sin(x)}{\cos(x)} = \frac{1}{\cos(x)} \cdot \sin(x)$

Let $u = \cos(x) \Rightarrow \frac{du}{dx} = -\sin(x) \Rightarrow dx = \frac{du}{-\sin(x)}$

$\Rightarrow \int \frac{\sin(x)}{\cos(x)}\,dx = \int \frac{\cancel{\sin(x)}}{\cos(x)} \cdot \frac{du}{-\cancel{\sin(x)}} = \int \frac{-1}{\cos(x)}\,du$ 2/ (a)

$= \int \frac{-1}{u}\,du$ 2/ (b)

3/

$\int \frac{-1}{u}\,du = -\int \frac{1}{u}\,du = -\ln|u| + C$

$\Rightarrow \int \tan(x)\,dx = -\ln|\cos(x)| + C$

2/ $\int x\sqrt{x+1}\,dx = ?$

Problem: No obvious choice of $g(x)$.

Leap of faith: $u = x+1$ ($x = u-1$) $\Rightarrow \frac{du}{dx} = 1 \Rightarrow du = dx$

$\Rightarrow \int x\sqrt{x+1}\,dx = \int x\sqrt{x+1}\,du = \int (u-1)\sqrt{u}\,du$

$= \int u^{3/2} - u^{1/2}\,du = \frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2} + C$

$= \frac{2}{5}(x+1)^{5/2} - \frac{2}{3}(x+1)^{3/2} + C$

Definite Version : $(F'(x) = f(x))$

$$\int f(g(x))\, g'(x)\, dx = F(g(x)) + C$$

$$\Rightarrow \int_a^b f(g(x))\, g'(x)\, dx = F(g(b)) - F(g(a))$$

$$= \int_{g(a)}^{g(b)} f(u)\, du$$

Not same in u-world.

General Advice : To calculate a definite integral always calculate the indefinite integral separately first.

Example

1/

$$\int_0^{\pi/4} \tan(x)\, dx = \, ?$$

Done above

$$\int \tan(x)\, dx = -\ln|\cos(x)| + C$$

$$\Rightarrow \int_0^{\pi/4} \tan(x)\, dx = -\ln|\cos(x)|\Big|_0^{\pi/4}$$

$$= (-\ln|\cos(\pi/4)|) - (-\ln|\cos(0)|)$$

$$= -\ln\left(\frac{1}{\sqrt{2}}\right)$$

Exception to Rule : If we can only calculate

$$\int_{g(a)}^{g(b)} f(u)\,du$$ geometrically.

Example

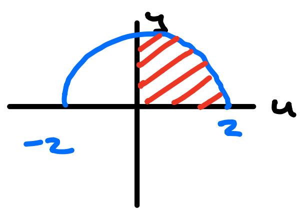
$$\int_0^{\sqrt{2}} \sqrt{4 - x^4} \cdot 2x \; dx = ?$$

Problem : No obvious choice of $g(x)$.

Leap of Faith : $u = x^2 \Rightarrow \dfrac{du}{dx} = 2x \Rightarrow dx = \dfrac{du}{2x}$

$$\Rightarrow \int_0^{\sqrt{2}} \sqrt{4 - x^4} \cdot 2x \; dx = \int_0^{2} \sqrt{4 - u^2}\; du$$

can't find antiderivative



top of semicircle radius 2 center $(0,0)$

$$\Rightarrow \int_0^{\sqrt{2}} \sqrt{4 - x^4} \cdot 2x \; dx = \text{Area}(\;) = \frac{1}{4} \cdot \pi \cdot 2^2 = \pi$$