Integration by Substitution

$\int f(x)\,dx$ = general antiderivative of $f(x)$

indefinite integral

$\int f(x)\,dx = F(x) + C$ (on interval)

particular antiderivative, ie $F'(x) = f(x)$

Recall Chain Rule : $\frac{d}{dx}\left(F(g(x))\right) = F'(g(x))\,g'(x)$

$\Rightarrow$ $F(g(x))$ = antiderivative of $F'(g(x))\,g'(x)$

$$\Rightarrow \quad \boxed{\int F'(g(x))\,g'(x)\,dx = F(g(x)) + C}$$

Example $\int 2x\cos(x^2)\,dx = ?$

Let $F(x) = \sin(x)$ and $g(x) = x^2$. Then $F'(x) = \cos(x)$ and $g'(x) = 2x$, hence $2x\cos(x^2) = F'(g(x))\,g'(x)$.

$$\Rightarrow \int 2x\cos(x^2)\,dx = \sin(x^2) + C$$

There is a way to formalize this to make it easier.

1/ Examine integrand. Is it of the form $f(g(x))\,g'(x)$ up to a constant? E.g. $\int 2e^{\sin(x)}\cos(x)\,dx$

2/ We are going to change the variable in the antiderivative to make it easier. Set $u = g(x)$. ← new variable

a) Replace dx with expression in du as follows:

$$u = g(x) \Rightarrow \frac{du}{dx} = g'(x) \Rightarrow dx = \frac{du}{g'(x)} \quad (\text{Just notation})$$

E.g. $u = \sin(x) \Rightarrow \frac{du}{dx} = \cos(x) \Rightarrow dx = \frac{du}{\cos(x)}$

b) Rewrite the new integrand purely in terms of u.

E.g. $$\int 2e^{\sin(x)}\cos(x)\,dx = \int 2e^{\sin(x)}\cos(x)\frac{du}{\cos(x)}$$

$$= \int 2e^{\sin(x)}\,du = \int 2e^{u}\,du$$

3/ Calculate indefinite integral in u variable. Finally replace u with $g(x)$.

E.g. $\int 2e^u\,du = 2\int e^u\,du = 2e^u + C = 2e^{\sin(x)} + C$

Here's why it works in general: $F'(x) = f(x)$, $u = g(x) \Rightarrow$

$\frac{du}{dx} = g'(x) \Rightarrow dx = \frac{du}{g'(x)} \Rightarrow \int f(g(x))\,g'(x)\,dx = \int f(g(x))\,g'(x)\cdot\frac{du}{g'(x)}$

$= \int f(g(x))\,du = \int f(u)\,du = F(u) + C = F(g(x)) + C$

This was what we got with Chain Rule.

<u>Conclusion</u>: $u = g(x) \Rightarrow \int f(g(x))\,g'(x)\,dx = \int f(u)\,du$

<u>Definite Version</u>: $\int_a^b f(g(x))\,g'(x)\,dx = F(g(x))\Big|_a^b$

$= F(g(b)) - F(g(a))$

$\int_{g(a)}^{g(b)} f(u)\,du = F(u)\Big|_{g(a)}^{g(b)} = F(g(b)) - F(g(a))$

$$\Rightarrow \boxed{\int_a^b f(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du}$$

Examples

1/ $\int x^3 \sin(x^4 + 2)\, dx = ?$

Let $u = x^4 + 2 \Rightarrow \frac{du}{dx} = 4x^3 \Rightarrow dx = \frac{du}{4x^3} \Rightarrow$

$$\int x^3 \sin(x^4+2)\, dx = \int x^3 \sin(x^4+2) \frac{du}{4x^3} = \frac{1}{4} \int \sin(x^4+2)\, du$$

$$= \frac{1}{4} \int \sin(u)\, du = \frac{-1}{4} \cos(u) + C = \frac{-1}{4} \cos(x^4+2) + C$$

2/ If $F'(x) = f(x)$ and a and b are constant

$\int f(ax+b)\, dx = ?$

Let $u = ax+b \Rightarrow \frac{du}{dx} = a \Rightarrow dx = \frac{du}{a} \Rightarrow$

$$\int f(ax+b)\, dx = \int f(ax+b) \cdot \frac{du}{a} = \underset{\text{constant}}{\frac{1}{a}} \int f(ax+b)\, du = \frac{1}{a} \int f(u)\, du$$

$$= \frac{1}{a} F(ax+b)$$

Conclusion: If $F'(x) = f(x) \Rightarrow \int f(ax+b)\, dx = \frac{1}{a} F(ax+b) + C$

<u>Warning</u> : We <u>cannot</u> apply this to more complicated compositions.

It only worked because $g'(x) = a$, a constant

E.g. $\displaystyle\int \cos(x^2)\,dx \neq \frac{\sin(x^2)}{2x} + C$

quotient rule messes up the derivative.

The substitution $u = g(x)$ will be useless for $\int f(g(x))\,dx$ in general.

3/ $\displaystyle\int_0^{\pi/4} \tan(x)\,dx = ?$ $\qquad \tan(x) = \frac{\sin(x)}{\cos(x)} = (\cos(x))^{-1} \cdot \sin(x)$

$u = \cos(x) \Rightarrow \dfrac{du}{dx} = -\sin(x) \Rightarrow dx = \dfrac{du}{-\sin(x)} \Rightarrow$

$$\int \tan(x)\,dx = \int \frac{\sin(x)}{\cos(x)} \cdot \frac{du}{-\sin(x)} = -\int \frac{1}{\cos(x)}\,du = -\int \frac{1}{u}\,du$$

$$= -\ln|u| + C = -\ln|\cos(x)| + C$$

$$\Rightarrow \int_0^{\pi/4} \tan(x)\,dx = -\ln|\cos(x)|\Big|_0^{\pi/4} = \left(-\ln|\cos(\pi/4)|\right) - \left(-\ln|\cos(0)|\right)$$

$$= -\ln\left(\frac{1}{\sqrt{2}}\right) - \left(-\ln(1)\right)$$

$$= \ln(\sqrt{2}) = \frac{\ln(2)}{2}$$

4/ (Hard) $\int x\sqrt{2x-1}\,dx = ?$

No obvious choice of u. Try $u = 2x-1 \Rightarrow \frac{du}{dx} = 2 \Rightarrow dx = \frac{du}{2}$

$$\Rightarrow \int x\sqrt{2x-1}\,dx = \frac{1}{2}\int x\sqrt{2x-1}\,du = \frac{1}{2}\int \frac{u-1}{2}\sqrt{u}\,du$$

$$= \frac{1}{4}\int u^{3/2}\,du - \frac{1}{4}\int u^{1/2}\,du = \frac{1}{4}\cdot\frac{1}{\left(\frac{5}{2}\right)}u^{5/2} - \frac{1}{4}\cdot\frac{1}{\left(\frac{3}{2}\right)}u^{3/2} + C$$

$$= \frac{1}{10}u^{5/2} - \frac{1}{6}u^{3/2} + C = \frac{1}{10}(2x-1)^{5/2} - \frac{1}{6}(2x-1)^{3/2} + C$$