The Precise Definition of a Limit

\[ \lim_{x \to a} f(x) = L \iff f(x) \text{ approaches } L \text{ as } x \text{ approaches (but does not equal) } a \text{ from both sides.} \]

Warm-Up Exercise:

a) Plot all points \((x, y)\) such that \(|y - 2| < 0.1\)

\[ |y - 2| < 0.1 \iff 2 - 0.1 < y < 2 + 0.1 \]
\[ 1.9 \quad 2.1 \]

b) Plot all points \((x, y)\) such that \(|x - 1| < 0.2\)

\[ |x - 1| < 0.2 \iff 1 - 0.2 < x < 1 + 0.2 \]
\[ 0.8 \quad 1.2 \]

Key Observation

Given any horizontal strip (centered at \(L\)), the graph \(y = f(x)\) is completely contained in the strip if \(x\) is close enough (but not equal) to \(a\).
Hence, given any horizontal strip (centered at $L$), there exists a vertical strip (centered at $a$) such that, within the vertical strip, the graph $y = f(x)$ is completely contained in the intersection.

Important: Many possible vertical strips will work (just lower the width). What matters is that at least one exists.
Non-example

\[ \lim_{x \to a} f(x) \neq L \]

Always outside intersection for any choice of vertical strip

Let's quantify this carefully.

\[ \epsilon > 0 \quad \text{in horizontal strip} \]
\[ \Rightarrow 0 < |x - a| < \delta \]
\[ \Rightarrow |f(x) - L| < \epsilon \]

Epsilon for "error"

\[ \delta > 0 \quad \text{in vertical strip} \]
\[ \Rightarrow \quad \quad \quad \quad \quad \quad \quad \quad \text{for all } x \text{ in the strip} \]
\[ \Rightarrow \quad \quad \quad \quad \quad \quad \quad \quad \text{for all } x \text{ in the strip} \]

Delta for "difference"
More Precise Definition of \( \lim_{x \to a} f(x) = L \)

Given any horizontal strip (centered at \( L \)), there exists a vertical strip (centered at \( a \)) such that

\[(x, t(x)) \text{ in vertical strip} \Rightarrow (x, t(x)) \text{ in horizontal strip} \quad \text{and} \quad x \neq a\]

Most Precise Definition of \( \lim_{x \to a} f(x) = L \)

A free choice of horizontal strip \quad A vertical strip

Given \( \varepsilon > 0 \), there exists \( \delta > 0 \) (which depends on \( \varepsilon \)) such that

\[0 < |x - a| < \delta \quad \Rightarrow \quad |f(x) - L| < \varepsilon\]
Important $\varepsilon > 0$ is a completely free choice. To be sure $\lim_{x \to a} f(x) = L$, we need to find a different $\delta > 0$ for each $\varepsilon > 0$.

Example $\lim_{x \to 1} 2x = 2$

Fix $\varepsilon > 0$

Try $\delta = \frac{\varepsilon}{2} > 0$

Need to be sure

$0 < |x - 1| < \frac{\varepsilon}{2} \Rightarrow |2x - 2| < \varepsilon$

But

$0 < |x - 1| < \frac{\varepsilon}{2} \Rightarrow 2|x - 1| < \varepsilon$

$\Rightarrow |2x - 2| < \varepsilon$

$\Rightarrow \lim_{x \to 1} 2x = 2$

Remarks

1. This example was relatively straightforward as $y = f(x)$ was a straight line. If not, it could be much more challenging. E.g. $\lim_{x \to 1} x^2 = 1$

2. Every limit law/property can be rigorously demonstrated using this $(\varepsilon, \delta)$ - language.

3. If we replace $0 < |x - a| < \delta$ with $a < x < a + \delta$ we get $\lim_{x \to a^+} f(x) = L$. 
If we replace $0 < |x-a| < \epsilon$ with $a-\delta < x < a$
we get $\lim_{x \to a^-} f(x) = L$

**Precise Definition of $\lim_{x \to a} f(x) = \infty$**

Given any $M > 0$, there exists $\delta > 0$ (which depends on $M$) such that

$0 < |x-a| < \delta \implies f(x) > M$

**Important:** $M > 0$ is as big as we want, hence $f(x)$ grows positively without bound.

**Remark**

We define $\lim_{x \to a} f(x) = -\infty$ replacing $M > 0$ with $N < 0$
and $f(x) > M$ with $f(x) < N$. 