The Precise Definition of a Limit

Mathematically vague

$\lim_{x \to a} f(x) = L \iff$ " $f(x)$ approaches L as x approaches (but does not equal) a from both sides."

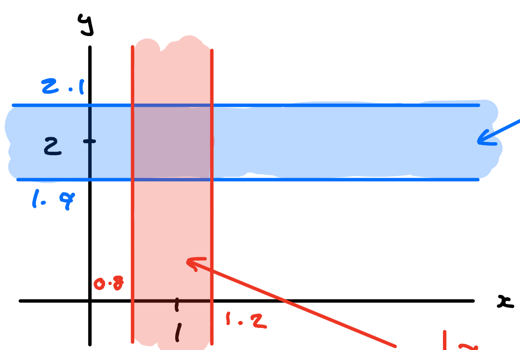
Warm-Up Exercise :

a) Plot all points (x,y) such that $|y-2| < 0.1$

b) Plot all points (x,y) such that $|x-1| < 0.2$

$$|y-2| < 0.1 \iff \underset{1.9}{\underset{\shortparallel}{2-0.1}} < y < \underset{2.1}{\underset{\shortparallel}{2+0.1}}$$

$$|x-1| < 0.2 \iff \underset{0.8}{\underset{\shortparallel}{1-0.2}} < x < \underset{1.2}{\underset{\shortparallel}{1+0.2}}$$



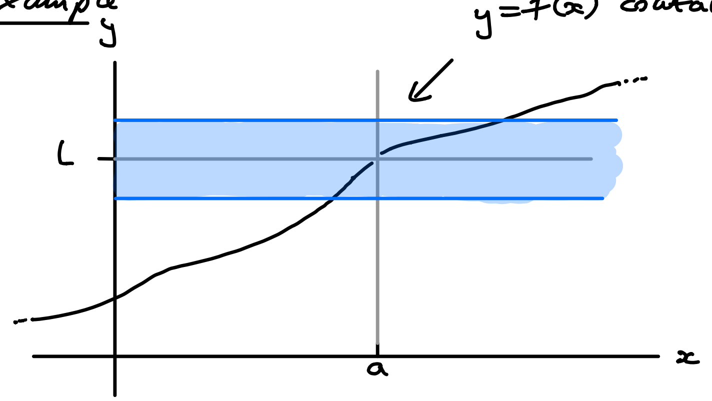
$|y-2| < 0.1$
Horizontal strip centered at 2

$|x-1| < 0.2$
Vertical strip centered at 1

Key Observation

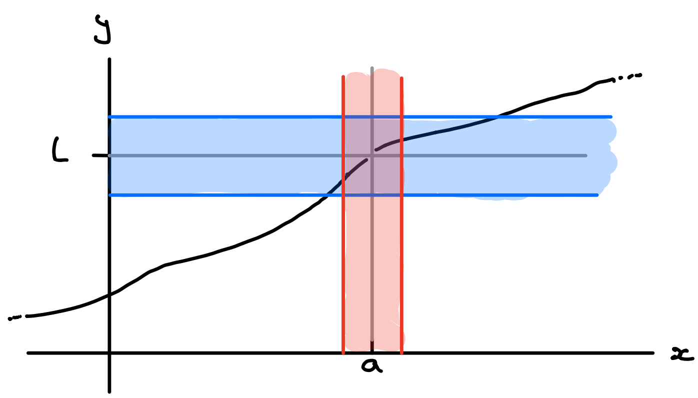
$\lim_{x \to a} f(x) = L \implies$ Given any horizontal strip (centered at L), the graph $y = f(x)$ is completely contained in the strip if x is close enough (but not equal) to a.

Example



Hence, given any horizontal strip (centered at L), there exists a vertical strip (centered at a) such that, within the vertical strip ($x \neq a$), the graph $y=f(x)$ is completely contained in the intersection.

Example

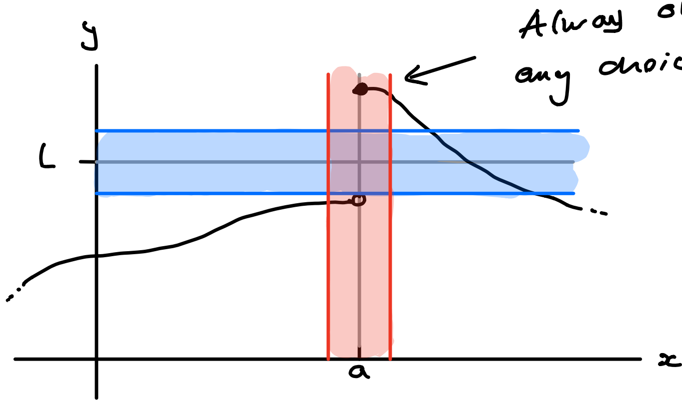


Important : Many possible vertical strips will work (just lower the width). What matters is that at least one exists.

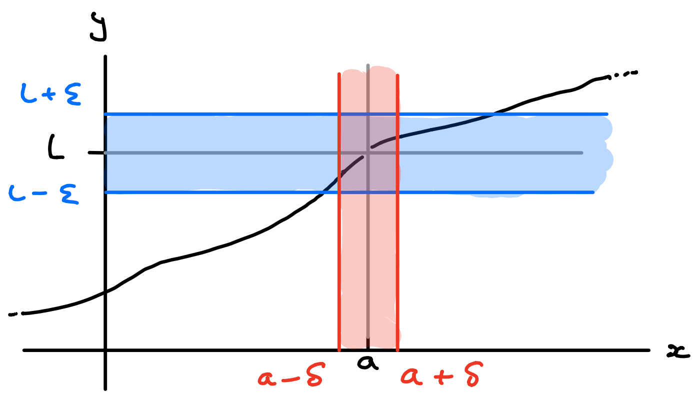
Non-example

$$\lim_{x \to a} f(x) \neq L$$

Always outside intersection for any choice of vertical strip



Let's quantify this carefully.



Epsilon for "error"

$\varepsilon > 0$

$(x, f(x))$ in horizontal strip

$\iff L - \varepsilon < f(x) < L + \varepsilon$

$\iff \quad |f(x) - L| < \varepsilon$

Delta for "difference"

$\delta > 0$

$(x, f(x))$ in vertical strip

$\iff a - \delta < x < a + \delta$

$\iff \quad |x - a| < \delta$

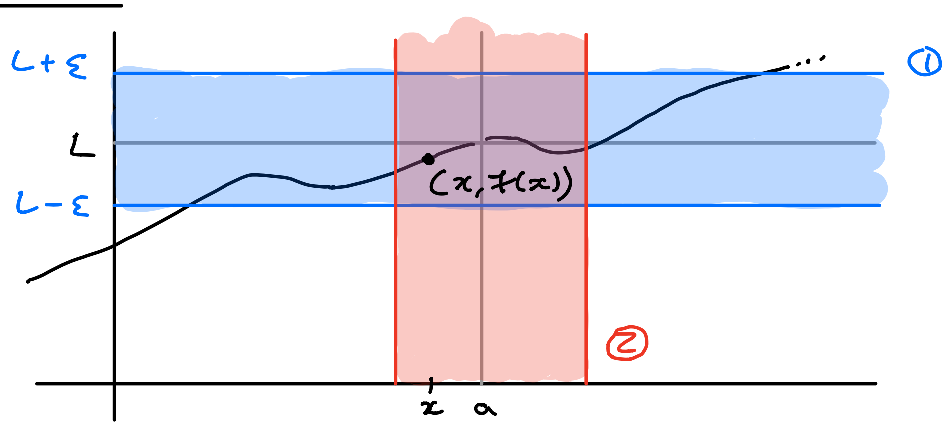
More Precise Definition of $\lim_{x \to a} f(x) = L$

Given any ① horizontal strip (centered at L), there exists a vertical strip ② (centered at a) such that

$(x, f(x))$ in vertical strip $\Rightarrow$ $(x, f(x))$ in horizontal strip

<u>and</u> $x \neq a$

<u>Picture</u>



Most Precise Definition of $\lim_{x \to a} f(x) = L$

a free choice of horizontal strip

a vertical strip

Given $\varepsilon > 0$, there exists $\delta > 0$ (which depends on ε) such that

$$0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$$

$(x, f(x))$ in vertical strip
$x \neq a$

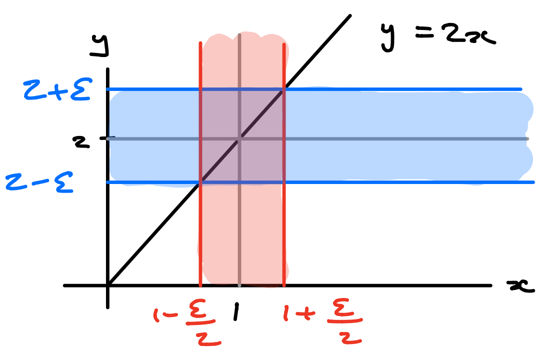
$(x, f(x))$ in horizontal strip

__Important__ $\varepsilon > 0$ is a __completely free choice__. To be sure $\lim_{x \to a} f(x) = L$ __we need to find a different $\delta > 0$__ __for each $\varepsilon > 0$.__

Could be really challenging

__Example__ $\lim_{x \to 1} 2x = 2$

Fix $\varepsilon > 0$



Try $\delta = \frac{\varepsilon}{2} > 0$

Need to be sure

$$0 < |x-1| < \frac{\varepsilon}{2} \Rightarrow |2x-2| < \varepsilon$$

But

$$0 < |x-1| < \frac{\varepsilon}{2} \Rightarrow 2|x-1| < \varepsilon$$

$$\Rightarrow |2(x-1)| < \varepsilon$$

$$\Rightarrow |2x-2| < \varepsilon$$

$$\Rightarrow \lim_{x \to 1} 2x = 2$$

__Remarks__

1) This example was relatively straightforward as $y = f(x)$ was a straight line. If not, it could be much more challenging. E.g. $\lim_{x \to 1} x^2 = 1$

2) Every limit law / property can be rigorously demonstrated using this (ε, δ) - language.

3) If we replace $0 < |x-a| < \delta$ with $a < x < a + \delta$ we get $\lim_{x \to a^+} f(x) = L$.

If we replace $0 < |x - a| < \delta$ with $a - \delta < x < a$

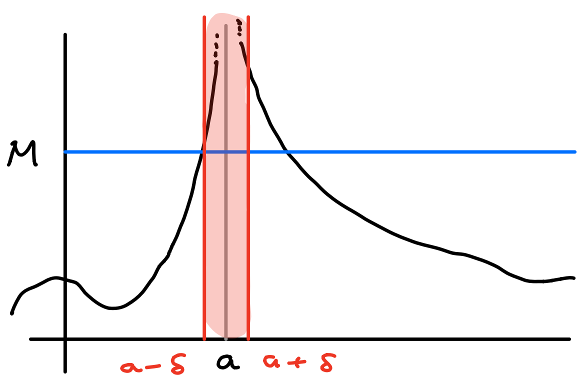
we get $\lim\limits_{x \to a^-} f(x) = L$

<u>Precise Definition of $\lim\limits_{x \to a} f(x) = \infty$</u>

Given any $M > 0$, there exists $\delta > 0$ (which depends on M)

such that

$0 < |x - a| < \delta \Rightarrow f(x) > M$ $\iff$ "for $(x, f(x))$ in vertical strip, $x \neq a$, $f(x) > M$"



<u>Important</u> : $M > 0$ is as big as we want, hence

$f(x)$ grows positively without bound.

<u>Remark</u>

We define $\lim\limits_{x \to a} f(x) = -\infty$ replacing $M > 0$ with $N < 0$

and $f(x) > M$ with $f(x) < N$.