Practice Problems

1/ Curve Sketching

Sketch $y = xe^{-1/x}$

2/ Optimization

What is the shortest possible line segment cut off by first quadrant which is tangent to $\frac{3}{x}$?

3/ Position and Velocity

A car is travelling at $30\ m/s$ when the driver sees an accident $40\ m$ ahead. What constant deceleration is needed to avoid a pile up?

4/ Riemann Sums

Calculate $\lim_{n \to \infty} \sum_{i=1}^{n} \frac{3n + 2i}{4n^2 + in}$.

5/ Volumes

A hole of radius r is bored through the center of a sphere of radius $R > r$. Find the volume of the remaining portion of the sphere.

1/ $f(x) = x e^{-1/x}$

<u>Domain</u> : $(-\infty, 0) \cup (0, \infty)$

<u>Odd/Even</u> : Neither

<u>Vertical Asymptotes</u> : $f(x) = \frac{x}{e^{1/x}} = \frac{e^{-1/x}}{1/x}$

$\lim_{x \to 0^+} \frac{-1}{x} = -\infty \Rightarrow \lim_{x \to 0^+} e^{-1/x} = 0 \Rightarrow \lim_{x \to 0^+} \frac{e^{-1/x}}{1/x} = 0$

$\lim_{x \to 0^-} \frac{-1}{x} = \infty \Rightarrow \lim_{x \to 0^-} e^{-1/x} = \infty$

$\Rightarrow \lim_{x \to 0^-} \frac{e^{-1/x}}{1/x} = \lim_{x \to 0^-} \frac{1/x^2 \, e^{-1/x}}{-1/x^2} = \lim_{x \to 0^-} -e^{-1/x} = -\infty$

$\Rightarrow$ $x = 0$ vertical asymptote

<u>Behavior at $\pm\infty$</u> : $\lim_{x \to \pm\infty} \frac{f(x)}{x} = \lim_{x \to \pm\infty} e^{-1/x} = 1$

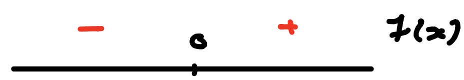
$\lim_{x \to \pm\infty} f(x) - x = \lim_{x \to \pm\infty} x(e^{-1/x} - 1) = \lim_{x \to \pm\infty} \frac{e^{-1/x} - 1}{1/x}$

L'Hospital $\longrightarrow$ $= \lim_{x \to \pm\infty} -e^{-1/x} = -1$

$\Rightarrow$ $y = x - 1$ slant asymptote

<u>Sign analysis on $f(x)$</u>

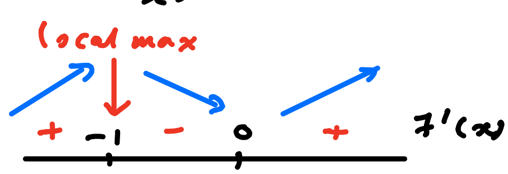
A/ $x e^{-1/x} = 0$ has no solutions in $\mathbb{R}$



B/ $x = 0$

$f'(x) = 1 \cdot e^{-1/x} + x \cdot \frac{1}{x^2} e^{-1/x} = \left(1 + \frac{1}{x}\right) e^{-1/x}$

A/ $\left(1 + \frac{1}{x}\right) e^{-1/x} = 0 \Leftrightarrow x = -1$



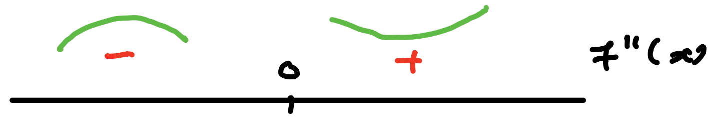
B/ $x = 0$

$f''(x) = \frac{-1}{x^2} e^{-1/x} + \left(1 + \frac{1}{x}\right) \cdot \frac{1}{x^2} e^{-1/x}$

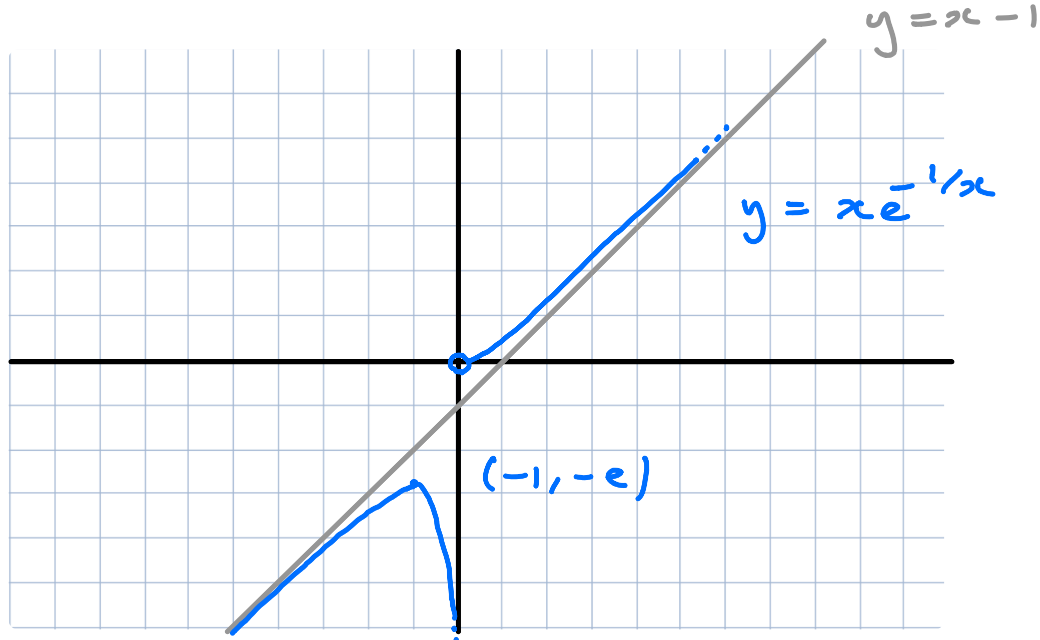
$= \left(-1 + \left(1 + \frac{1}{x}\right)\right) \frac{1}{x^2} e^{-1/x} = \frac{1}{x^3} e^{-1/x}$

A/ None

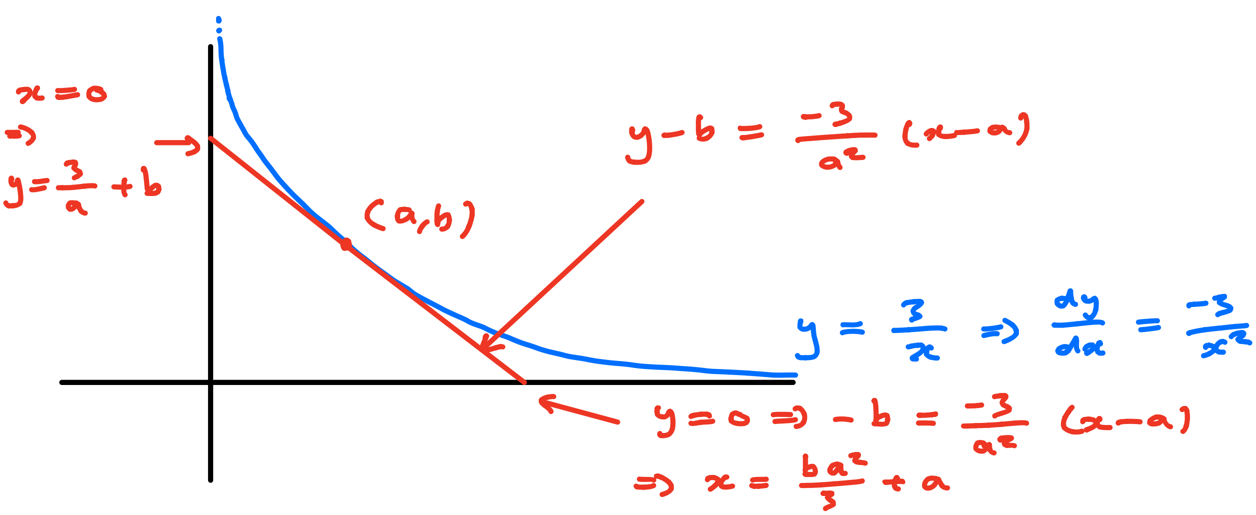
B/ $x = 0$



$f(-1) = (-1)e = -e$



2/ Objective Quantity : Minimize length of line



Objective Quantity $= \sqrt{\left(\frac{3}{a}+b\right)^2 + \left(\frac{ba^2}{3}+a\right)^2}$

Constraint : $b = \frac{3}{a}$, $a > 0$

To minimize root
minimize inside

$\Rightarrow$

$$\sqrt{\left(\frac{3}{a}+b\right)^2 + \left(\frac{ba^2}{3}+a\right)^2} = \sqrt{\left(\frac{3}{a}+\frac{3}{a}\right)^2 + (a+a)^2}$$

$$f(a) = \left(\frac{6}{a}\right)^2 + (2a)^2 = \frac{36}{a^2} + 4a^2$$

Domain : $(0, \infty)$

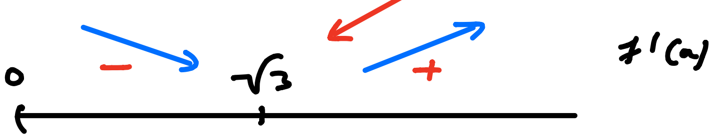
$$f'(a) = \frac{-72}{a^3} + 8a$$

A/ $\frac{-72}{a^3} + 8a = 0 \Rightarrow 8a = \frac{72}{a^3} \Rightarrow a^4 = 9$

$\Rightarrow a = \sqrt{3}$ in $(0, \infty)$

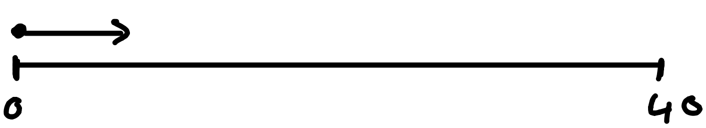
B/ None in $(0, \infty)$

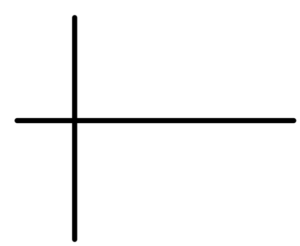
f cts at $\sqrt{3}$ $\Rightarrow$ $f(\sqrt{3})$ Abs. min



$\Rightarrow$ shortest line segment $= \sqrt{f(\sqrt{3})} = \sqrt{\frac{36}{3} + 4\cdot 3} = \sqrt{24}$

3/ $s(0) = 0$
$v(0) = 30$





$a(t) = k \Rightarrow v(t) = kt + C_1$

$v(0) = 30 \Rightarrow C_1 = 30 \Rightarrow v(t) = kt + 30$

$v(t) = 0 \Rightarrow kt + 30 = 0 \Rightarrow t = \frac{-30}{k}$

time when car stops

$\Rightarrow s(t) = \frac{k}{2}t^2 + 30t + C_2$

$s(0) = 0 \Rightarrow C_2 = 0 \Rightarrow s(t) = \frac{k}{2}t^2 + 30t$

position when car stops

$$s\left(\frac{-30}{k}\right) = \frac{k}{2}\left(\frac{-30}{k}\right)^2 + 30 \cdot \left(\frac{-30}{k}\right)$$

$$= \frac{-450}{k}$$

$$s\left(\frac{-30}{k}\right) = 40 \Rightarrow \frac{-450}{k} = 40 \Rightarrow k = \frac{-450}{40}$$

$\Rightarrow$ Car must decelerate at $\frac{450}{40}$ m/s^2 at minimum.

4/ $$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{3n+2i}{4n^2+in} = ?$$

Need a, b, f such that

$$\frac{3n+2i}{4n^2+in} = f(\underbrace{x_i}_{= a + \frac{b-a}{n} i}) \underbrace{\Delta x}_{= \frac{b-a}{n}}$$

Simplifying assumptions : $a = 0, b = 1 \Rightarrow \Delta x = \frac{1}{n}$, $x_i = \frac{i}{n}$

just need to choose appropriate $f(x)$

$$\frac{3n+2i}{4n^2+in} = \underbrace{\frac{3n+2i}{4n+i}}_{f\left(\frac{i}{n}\right)} \cdot \overbrace{\frac{1}{n}}^{\Delta x}$$

$$\frac{3n+2i}{4n+i} = \frac{3+2\left(\frac{i}{n}\right)}{4+\left(\frac{i}{n}\right)} = f\left(\frac{i}{n}\right) \Rightarrow f(x) = \frac{3+2x}{4+x}$$

$$\Rightarrow \lim_{n\to\infty} \sum_{i=1}^{n} \frac{3n+2i}{4n^2+in} = \int_0^1 \frac{3+2x}{4+x}\,dx$$

$x = u-4$

$$u = 4+x \Rightarrow \frac{du}{dx} = 1 \Rightarrow dx = du \Rightarrow$$

$$\int \frac{3+2x}{4+x}\,dx = \int \frac{3+2x}{4+x}\,dx = \int \frac{3+2(u-4)}{u}\,du$$

$$= \int 2 + \frac{-5}{u}\,du = 2u - 5\ln|u| + C$$

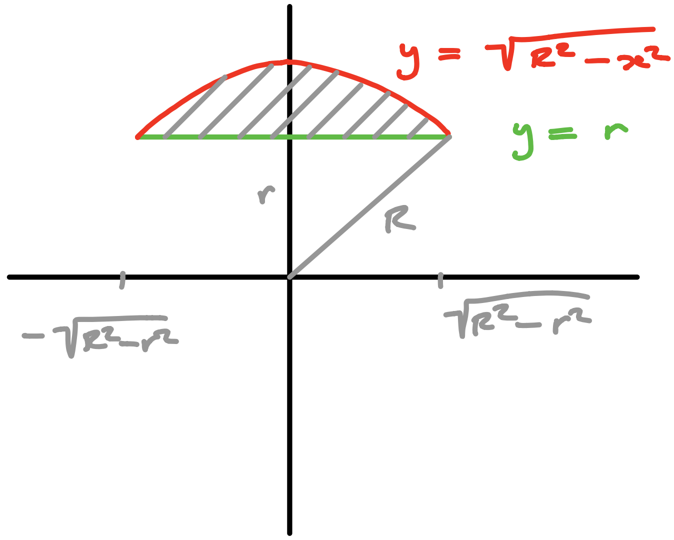
$$= 2(x+4) - 5\ln|x+4| + C$$

$$\Rightarrow \int_0^1 \frac{3+2x}{4+x}\,dx = 2(x+4) - 5\ln|x+4| \Big|_0^1$$

$$= 2 - 5(\ln(5) - \ln(4))$$

$$\Rightarrow \lim_{n\to\infty} \sum_{i=1}^{n} \frac{3n+2i}{4n^2+in} = 2 - 5(\ln(5) - \ln(4))$$

5/



$$\text{Volume} = \int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}} \pi\left((R^2-x^2) - r^2\right)\, dx$$

$$= \pi(R^2-r^2)x - \frac{\pi}{3}x^3 \Bigg|_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}}$$

$$= 2\pi(R^2-r^2)\sqrt{R^2-r^2} - \frac{2\pi}{3}\left(\sqrt{R^2-r^2}\right)^3$$

$$= \left(2\pi - \frac{2\pi}{3}\right)(R^2-r^2)^{3/2}$$

$$= \frac{4\pi}{3}(R^2-r^2)^{3/2}$$