Derivatives of Power and Exponential Functions

Recall: $f'(x) = \frac{d}{dx}(f(x)) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$ = Slope of tangent line at $(x, f(x))$

Power Functions

$f(x) = x^n$, n positive integer.

$n=0 \Rightarrow f(x) = x^0 = 1 \Rightarrow f'(x) = \lim_{h\to 0}\frac{1-1}{h} = \lim_{h\to 0} 0 = 0$

$n \geq 1 \Rightarrow f'(x) = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}$

How can we expand $(x+h)^n$?

Pascal's Triangle:

0^{th} Row $\longrightarrow$ 1

1^{st} Row $\longrightarrow$ 1 1

2^{nd} Row $\longrightarrow$ 1 2 1

3^{rd} Row $\longrightarrow$ 1 3 3 1

4^{th} Row $\longrightarrow$ 1 4 6 4 1

5^{th} Row $\longrightarrow$ 1 5 10 10 5 1

$\vdots$ $\quad \binom{5}{2} \quad \binom{5}{3} \quad \ddots$

0^{th} Position, 1^{st} Position, 2^{nd}, 3^{rd}, 4^{th}, 5^{th}

Just notation

Notation $0 \leq k \leq n$, $\binom{n}{k}$ = Value in the k^{th} position of the n^{th} row.

Facts

1/ $\binom{n}{k}$ = number of ways of choosing k unordered objects from a collection of n objects.

Example $\binom{n}{1} = n$

2/ $$\binom{n}{k} = \frac{n \times (n-1) \times (n-2) \times \dots \times (n-(k-1))}{k \times (k-1) \times (k-2) \times \dots \times 2 \times 1}$$

$$= \frac{n!}{k!\,(n-k)!}$$ (By convention $0! = 1$)

Example

$$\binom{n}{1} = \frac{n!}{1!\,(n-1)!} = \frac{n \times \cancel{(n-1)} \times \dots \times \cancel{2} \times \cancel{1}}{\cancel{(n-1)} \times \cancel{(n-2)} \times \dots \times \cancel{1}} = n$$

$$\binom{5}{3} = \frac{5!}{3!\,2!} = \frac{5 \times 4 \times \cancel{3} \times \cancel{2} \times \cancel{1}}{\cancel{(3 \times 2 \times 1)} \times (2 \times 1)} = \frac{20}{2} = 10$$

Binomial Theorem

$$(x+h)^n = x^n + \binom{n}{1} x^{n-1} h + \binom{n}{2} x^{n-2} h^2 + \dots + \binom{n}{n-1} x h^{n-1} + h^n$$

0^{th} Row $\longrightarrow$ 1

1^{st} Row $\longrightarrow$ 1 1 $\longrightarrow$ $(x+h)^1 = x + h$

2^{nd} Row $\longrightarrow$ 1 2 1 $\longrightarrow$ $(x+h)^2 = x^2 + 2xh + h^2$

3^{rd} Row $\longrightarrow$ 1 3 3 1 $\longrightarrow$ $(x+h)^3 = x^3 + 3x^2h + 3xh^2 + h^3$

4^{th} Row $\longrightarrow$ 1 4 6 4 1 $\longrightarrow$ $(x+h)^4 = x^4 + 4x^3h + 6x^2h^2 + 4xh^3 + h^4$

5^{th} Row $\longrightarrow$ 1 5 10 10 5 1 $\rightarrow$ $(x+h)^5 = x^5 + 5x^4h + 10x^3h^2 + 10x^2h^3 + 5xh^4 + h^5$

$\vdots$

$f(x) = x^n$, n positive integer, $n \geq 1$

$$\Rightarrow f'(x) = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{x^n} + \binom{n}{1} x^{n-1} h + \binom{n}{2} x^{n-2} h^2 + \dots + \binom{n}{n-1} x h^{n-1} + h^n - \cancel{x^n}}{h}$$

$$= \lim_{h \to 0} \binom{n}{1} x^{n-1} + \binom{n}{2} x^{n-2} h + \cdots + h^{n-1}$$

$$= \binom{n}{1} x^{n-1} = n x^{n-1}$$

<u>General Power Rule</u> For r any real number

$$\frac{d}{dx}(x^r) = r x^{r-1}$$

<u>Examples</u>

1/ $\frac{d}{dx}(x^2) = 2 \cdot x^{2-1} = 2x$

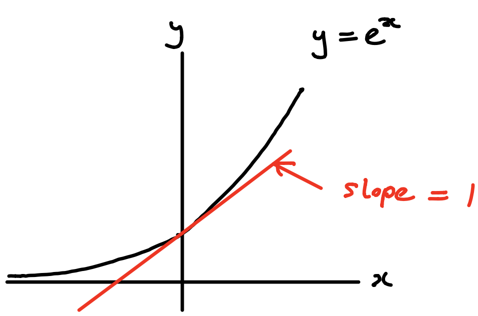
2/ $\frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{\frac{1}{2}}) = \frac{1}{2} x^{\frac{1}{2}-1} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

3/ $\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1}) = -1\, x^{-1-1} = \frac{-1}{x^2}$

<u>Exponential Functions</u>

Recall : $e = 2.71828\ldots$ is specifically chosen so that

slope of tangent line to $y = e^x$ at $x = 0$ $= 1$



$$\Rightarrow \frac{d}{dx}(e^x)\Big|_{x=1} = \lim_{h \to 0} \frac{e^{0+h} - e^0}{h}$$

$$\|$$

Defining Property of e $\longrightarrow$ $\boxed{\lim_{h \to 0} \frac{e^h - 1}{h} = 1}$

<u>More generally</u> : $\lim_{h \to 0} \frac{b^h - 1}{h}$ = Slope of tangent line to $y = b^x$ at $x = 0$

Observation

$$\frac{d}{dx}(e^x) = \lim_{h \to 0} \frac{e^{(x+h)} - e^x}{h} = \lim_{h \to 0} \frac{e^x \cdot e^h - e^x \cdot 1}{h}$$

Constant Multiple Law

$$= \lim_{h \to 0} e^x \cdot \frac{e^h - 1}{h} = e^x \lim_{h \to 0} \frac{e^h - 1}{h} = e^x \cdot 1 = e^x$$

<u>Important Conclusion</u> : $\boxed{\frac{d}{dx}(e^x) = e^x}$

More generally : $\frac{d}{dx}(b^x) = b^x \cdot \left(\lim_{h \to 0} \frac{b^h - 1}{h}\right)$

$= b^x \cdot$ (slope of tangent line to $y = b^x$ at $x = 0$)

Derivative Rules

<u>Sum Rule</u> : $\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))$

constant

<u>Constant Multiple Rule</u> : $\frac{d}{dx}(c f(x)) = c \frac{d}{dx}(f(x))$

Sum + CMR $\Rightarrow$ <u>Difference Rule</u>

$$\frac{d}{dx}(f(x) - g(x)) = \frac{d}{dx}(f(x)) - \frac{d}{dx}(g(x))$$

Proof of Sum

$$\frac{d}{dx}(f(x) + g(x)) = \lim_{h \to 0} \frac{(f(x+h) + g(x+h)) - (f(x) + g(x))}{h}$$

$$= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h}\right)$$

Sum Limit Law $\quad = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} + \lim_{h\to 0}\frac{g(x+h)-g(x)}{h}$

$$= \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))$$

Examples

1/ Find all points on $y = e^x - x$ such that tangent line is perpendicular to $y = 6x + 7$

$$\frac{dy}{dx} = \frac{d}{dx}(e^x - x) = \frac{d}{dx}(e^x) - \frac{d}{dx}(x') = e^x - 1$$

Tangent perpendicular to $y = 6x+7 \iff$ slope $= \frac{-1}{6}$

$$e^x - 1 = \frac{-1}{6} \iff e^x = \frac{5}{6} \iff x = \ln\left(\frac{5}{6}\right)$$

$\Rightarrow$ Only point is $\left(\ln\left(\frac{5}{6}\right), e^{\ln\left(\frac{5}{6}\right)} - \ln\left(\frac{5}{6}\right)\right)$

$$= \left(\ln\left(\frac{5}{6}\right), \frac{5}{6} - \ln\left(\frac{5}{6}\right)\right)$$

2/ Aim : Calculate $\frac{d}{dx}(b^x)$

Fact : $\boxed{\lim_{z\to 0}\frac{e^z - 1}{z} = 1}$

A/ $\lim_{h\to 0}\frac{e^{2h} - 1}{h} = ?$

$$\lim_{h\to 0} \frac{e^{2h}-1}{h} = \lim_{h\to 0} 2 \cdot \frac{e^{2h}-1}{2h} = 2 \lim_{h\to 0} \frac{e^{2h}-1}{2h} = 2\cdot 1 = 2$$

B/ $\lim_{h\to 0} \frac{e^{ah}-1}{h} = ?$ $(a \neq 0)$

$$\lim_{h\to 0} \frac{e^{ah}-1}{h} = \lim_{h\to 0} a \cdot \frac{e^{h}-1}{ah} = a \lim_{h\to 0} \frac{e^{ah}-1}{ah} = a\cdot 1 = a$$

C/ $\lim_{h\to 0} \frac{b^{h}-1}{h} = ?$

Recall $b = e^{\ln(b)} \Rightarrow b^{h} = e^{\ln(b)h}$

B/ $\Rightarrow$ $\underline{\lim_{h\to 0} \frac{b^{h}-1}{h}} = \lim_{h\to 0} \frac{e^{\ln(b)h}-1}{h} = \underline{\ln(b)}$

D/ $\frac{d}{dx}(b^x) = b^x \cdot \lim_{h\to 0} \frac{b^{h}-1}{h} = b^x \cdot \ln(b)$

<u>Conclusion</u> : $\boxed{b \neq 1 \Rightarrow \frac{d}{dx}(b^x) = b^x \cdot \ln(b)}$