**Optimization**

Motivating Example: What is the minimum value the sum of two positive numbers can be, if their product is 100?

Quantity unknowns: \(x, y > 0\)

Identity objective quantity to be minimized / maximized: \(x + y\)

Identity constraint: \(xy = 100\)

Solve constraint in one unknown and substitute into objective:

\[ xy = 100 \Rightarrow y = \frac{100}{x} \]

\[ x + y = x + \frac{100}{x} = f(x) \]

Identity domain and find absolute extrema with appropriate technique:

\(x, y > 0\) and \(xy = 100 \Rightarrow x > 0\)

Need to find absolute min of \(f(x) = x + \frac{100}{x}\) on \((0, \infty)\)

\((0, \infty) \neq \text{closed interval} \Rightarrow \text{Must use critical point theorem / global sign analysis of } f'(x).\)

\[ f'(x) = 1 - \frac{100}{x^2} \]

\[ f'(x) = 0 \Rightarrow 1 - \frac{100}{x^2} = 0 \Rightarrow x^2 = 100 \Rightarrow x = \pm 10 \]

By \(f'\) undefined when \(x = 0\)

\[ f'(10) = -99 < 0 \quad f'(100) = 1 - \frac{1}{100} = \frac{99}{100} > 0 \]

\(x = 10 \) is an absolute minimum at \( f(x) \) on \((0, \infty)\).

\[ x = 10, \quad y = \frac{100}{x} \Rightarrow y = 10 \]

Gather it all together into single conclusion:

If \(x, y > 0\) and \(xy = 100\) then the minimum value of \(x + y\) is 20 when \(x = 10\) and \(y = 10\).
Remark: A problem like this is called a constrained optimization problem.

Useful Fact:

First Derivative Test for Absolute Max/Min:

Let \( f \) be a continuous function on interval \( I \). Assume there is a single critical number \( c \) in \( I \) (excluding endpoints). Then

\[
\begin{align*}
+ & \quad \Rightarrow \quad f(c) \text{ Absolute Max} \\
- & \quad \Rightarrow \quad f(c) \text{ Absolute Min}
\end{align*}
\]

Strategy to Solve Constrained Optimization Problems:

1. Read problem very carefully. Identify what objective quantity you are being asked to maximize/minimize.
2. Perhaps draw a picture and label unknowns (generally there will be two degrees of freedom in this course).
3. Express objective quantity in terms of unknowns.
4. Identify what constraint is imposed by problem. Express as constraint equation in unknowns.
5. Solve constraint equation in one variable and substitute into objective to get a single variable function \( f \).
6. Identify domain imposed by \( f \) and nature of problem and find absolute max/min.
7. Conclude by explicitly stating the solution to the problem.
Example: A coffee company wants to manufacture cylindrical coffee cans with a volume of 1000 cm³. What should the radius and height be to minimize surface area?

We are trying to minimize surface area of a cylinder.

1. Surface area = Area of top and bottom + Area of sides
   = \( \pi r^2 + \pi r^2 + 2\pi rh \)
   = \( 2\pi r^2 + 2\pi rh \)

2. Constraint: Volume is 1000
   Volume = (area of base) \times \text{height}
   = \( \pi r^2 \times h \)
   \[ \Rightarrow \pi r^2 h = 1000 \]

3. Solve constraint in \( h \):
   \( \pi r^2 h = 1000 \)
   \[ \Rightarrow h = \frac{1000}{\pi r^2} \]

Substitute into objective:
\[ 2\pi r^2 + 2\pi h = 2\pi r^2 + 2\pi \frac{1000}{\pi r^2} = 2\pi r^2 + \frac{2000}{r} = f(r) \]

4. \( r, h \geq 0 \) and \( \pi r^2 h = 1000 \) \( \Rightarrow \) \( r > 0 \)

Need to minimize \( f(r) = 2\pi r^2 + \frac{2000}{r} \) on \((0, \infty)\)

\[ f'(r) = 4\pi r - \frac{2000}{r^2} \]
\[ A/ \quad f'(r) = 0 \Rightarrow 4\pi r - \frac{2000}{r^2} = 0 \Rightarrow 4\pi r = \frac{2000}{r^2} \]
\[ = r^3 = \frac{500}{\pi} \Rightarrow r = \left( \frac{500}{\pi} \right)^{\frac{1}{3}} \approx 5.419 \]

\[ B/ \quad f'(r) \text{ undetermined} \Rightarrow r = 0 \]

\[ y = x^2 \]

\[ y = 2, h = \frac{1000}{\pi r^2} \Rightarrow h = \frac{1000}{\pi \left( \frac{500}{\pi} \right)^{\frac{1}{3}}} \]

\[ C/ \quad \text{The surface area is at a minimum when} \]
\[ r = \left( \frac{500}{\pi} \right)^{\frac{1}{3}} \text{ and } h = \frac{1000}{\pi \left( \frac{500}{\pi} \right)^{\frac{1}{3}}} \]

\[ \text{Example} \]

Determine the point on the curve \( y = x^2 \) closest to \((2, \frac{1}{2})\).

1) We are trying to minimize the distance between \( y = x^2 \) and \((2, \frac{1}{2})\).

2) \[ y = x^2 \]

3) Distance between \((x, y)\) and \((2, 1)\) = \[ \sqrt{(x-2)^2 + (y-1/2)^2} \]

4) Constraint: \( y = x^2 \) \( \text{ No restrictions on } x \).

5) \( y = x^2 \) \( \Rightarrow \sqrt{(x-2)^2 + (y-1/2)^2} = \sqrt{(x-2)^2 + (x^2 - 1/2)^2} = \frac{f(x)}{} \)
Need to minimize \( f(x) \) on \( \mathbb{R} \).

\[
f'(x) = \frac{1}{2\sqrt{(x-z)^2 + (x^2 - \frac{1}{2})^2}} \cdot \left( 2(x-z) + 2(x^2 - \frac{1}{2}) \cdot 2x \right)
\]

\[
= \frac{4x^3 - 4}{2\sqrt{(x-z)^2 + (x^2 - \frac{1}{2})^2}}
\]

A. \( f'(x) = 0 \Rightarrow 4x^3 - 4 = 0 \Rightarrow x = 1 \)

B. \( \sqrt{(x-z)^2 + (x^2 - \frac{1}{2})^2} = \text{distance between } (x, y) \text{ on } y = x^2 \) and \( (2, \frac{1}{2}) \)

\( \Rightarrow \text{No } y \text{ points} \Rightarrow f'(x) \text{ is abs min} \)

\[
\begin{array}{c|c|c}
& 0 & 2 \\
\hline
f'(x) & < 0 & > 0 \\
\end{array}
\]

C. \( x = 1 \Rightarrow y = 1^2 = 1 \Rightarrow (1, 1) \text{ is closest point to } (2, \frac{1}{2}) \text{ on } y = x^2. \)

Example: Minimize \( (x+z)^2 + y^2 \) subject to constraint \( 5y + z = 5 \) and \( x, y \geq 0 \)

Objective: Minimize \( (x+z)^2 + y^2 \)

Constraint: \( 5y + z = 5 \) and \( x, y \geq 0 \)

\[
y = 1 - \frac{z}{5} \Rightarrow (x+z)^2 + (1 - \frac{z}{5})^2 = f(x, y)
\]

\[
\begin{array}{c|c}
y = 1 - \frac{z}{5} & \Rightarrow \text{ Domain } = [0, 5]
\end{array}
\]
\[ f'(x) = 2(x+1) - \frac{x}{5} (1- \frac{x}{5}) = \frac{5x^2}{5} x + \frac{18}{5} \]

By \[ f'(x) = 0 \Rightarrow x = \frac{-18}{5} \left( \frac{5x}{5x} \right) \]

\[ < 0 \Rightarrow \text{No type A critical numbers in } [0,5] \]

By Endpoints 0, 5

\[ f(0) = 5 \quad \text{minimal value.} \]

\[ f(5) = 49 \]

Warning: If we have calculated \[ f(\frac{-18}{5}) \left( \frac{5x}{5x} \right) \]

get a smaller value. It's just not in domain. Be very careful to consider correct domain.