Optimization

Motivating Example: What is the minimum value the sum of two positive numbers can be, if their product is 100?

Quantity unknowns: $x, y \geq 0$

Identify objective quantity to be minimized / maximized: $x + y$

Identify constraint: $xy = 100$

Solve constraint in one unknown and substitute into objective:

$$xy = 100 \Rightarrow y = \frac{100}{x}$$

$$\Rightarrow x + y = x + \frac{100}{x} = f(x)$$

Identify domain and find absolute extrema with appropriate technique:

$x, y \geq 0$ and $xy = 100 \Rightarrow x > 0$

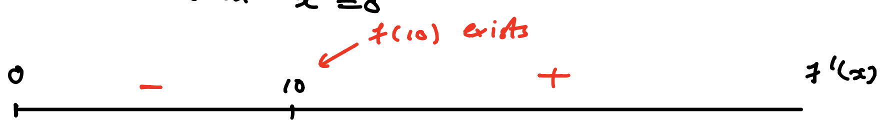
Need to find absolute min of $f(x) = x + \frac{100}{x}$ on $(0, \infty)$

$(0, \infty) \neq$ closed interval $\Rightarrow$ Must use critical point theorem / global sign analysis of $f'(x)$.

$$f'(x) = 1 - \frac{100}{x^2}$$

A/ $f'(x) = 0 \Rightarrow 1 - \frac{100}{x^2} = 0 \Rightarrow x^2 = 100 \Rightarrow x = \pm 10$

B/ f' undefined when $x = 0$



$f'(1) = -99 < 0$ $\qquad$ $f'(100) = 1 - \frac{1}{100} = \frac{99}{100} > 0$

$\Rightarrow f(10) = 10 + \frac{100}{10} = 20$ is an absolute minimum of $f(x)$ on $(0, \infty)$. $\left(x = 10,\ y = \frac{100}{x} \Rightarrow y = 10 \right)$

Gather it all together into single conclusion:

If $x, y \geq 0$ and $xy = 100$ then the minimum value of $x + y$ is 20 when $x = 10$ and $y = 10$.

Remark : A problem like this is called a constrained optimization problem.

Useful Fact :

First Derivative Test for Absolute Max/Mins

f - continuous function on interval. Assume there is a single critical number in I (excluding endpoints). Then

$\underline{+ \quad c \quad -}$ $f'(x)$ $\Rightarrow$ $f(c)$ Absolute Max

$\underline{- \quad c \quad +}$ $f'(x)$ $\Rightarrow$ $f(c)$ Absolute Min

Strategy to Solve Constrained Optimization Problems

1/ Read problem very carefully. Identify what objective quantity you are being asked to maximize / minimize.

2/ Perhaps draw a picture and label unknowns (generally there will be two degrees of freedom in this course)

3/ Express objective quantity in terms of unknowns.

4/ Identify what constraint is imposed by problem. Express as constraint equation in unknowns.

5/ Solve constraint equation in one variable and substitute into objective to get a single variable function f.

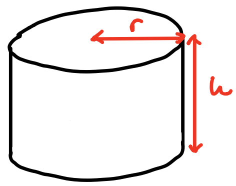
6/ Identify domain imposed by f and nature of problem and find absolute max/min.

7/ Conclude by explicitly stating the solution to the problem.

<u>Example</u> A coffee company wants to manufacture cylindrical coffee cans with a volume of 1000 cm^3. What should the radius and height be to minimize surface area.

1/ We are trying to minimize surface area of a cylinder

2/



3/ Surface area = Area of top and bottom + Area of sides

$$= \pi r^2 + \pi r^2 + \underbrace{2\pi r}_{\text{circumference}} \times h = \underbrace{2\pi r^2 + 2\pi r h}_{\text{Objective}}$$

4/ Constraint : Volume is 1000

Volume = (area of base) × height

$$= \pi r^2 \cdot h$$

$$\Rightarrow \quad \pi r^2 h = 1000$$

5/ Solve constraint in h : $\pi r^2 h = 1000$

$$\Rightarrow \quad h = \frac{1000}{\pi r^2}$$

Substitute into objective :

$$2\pi r^2 + 2\pi h = 2\pi r^2 + 2\pi r \cdot \frac{1000}{\pi r^2} = 2\pi r^2 + \frac{2000}{r} = f(r)$$

6/ $r, h \geq 0$ and $\pi r^2 h = 1000 \Rightarrow r > 0$

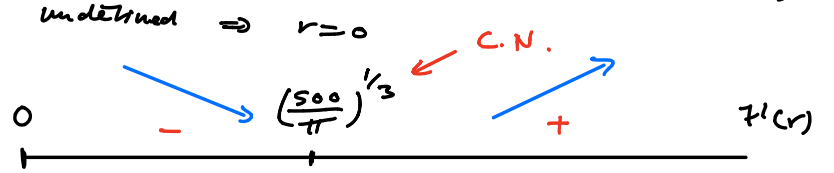
Need to minimize $f(r) = 2\pi r^2 + \frac{2000}{r}$ on $(0, \infty)$

$$f'(r) = 4\pi r - \frac{2000}{r^2}$$

A/ $f'(r) = 0 \Rightarrow 4\pi r - \frac{2000}{r^2} = 0 \Rightarrow 4\pi r = \frac{2000}{r^2}$

$\Rightarrow r^3 = \frac{500}{\pi} \Rightarrow r = \left(\frac{500}{\pi}\right)^{\frac{1}{3}} \approx 5.419$

B/ f' undefined $\Rightarrow r = 0$



$f'(1) = 4\pi - 2000 < 0$ $\quad f'(100) = 400\pi - \frac{2000}{10000} > 0$

$\Rightarrow f\left(\left(\frac{500}{\pi}\right)^{1/3}\right)$ absolute min of f on $(0, \infty)$

$r = \left(\frac{500}{\pi}\right)^{1/3}, \quad h = \frac{1000}{\pi r^2} \Rightarrow h = \frac{1000}{\pi \left(\frac{500}{\pi}\right)^{2/3}}$

7/ The surface area is at a minimum when

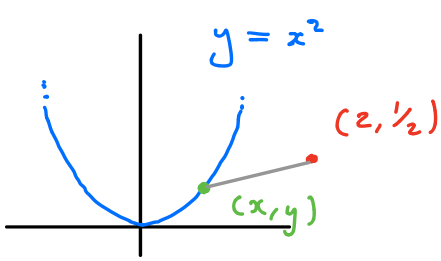
$r = \left(\frac{500}{\pi}\right)^{1/3}$ and $h = \frac{1000}{\pi \left(\frac{500}{\pi}\right)^{2/3}}$

Example

Determine the point on the curve $y = x^2$ closest to $(2, 1/2)$.

1/ We are trying to minimize the distance between $y = x^2$ and $(2, 1/2)$

2/



Pythagoras

3/ Distance between (x, y) and $(2, 1)$ $= \underbrace{\sqrt{(x-2)^2 + (y - 1/2)^2}}_{\text{Objective}}$

4/ Constraint : $y = x^2$ $\longleftarrow$ No restrictions on x.

5/ $y = x^2 \Rightarrow \sqrt{(x-2)^2 + (y - \frac{1}{2})^2} = \sqrt{(x-2)^2 + (x^2 - \frac{1}{2})^2} = f(x)$

6/ Need to minimize $f(x)$ on $\mathbb{R}$.

$$f'(x) = \frac{1}{2\sqrt{(x-2)^2 + (x^2-\frac{1}{2})^2}} \cdot \left(2(x-2) + 2(x^2-\tfrac{1}{2}) \cdot 2x\right)$$

$$= \frac{4x^3 - 4}{2\sqrt{(x-2)^2 + (x^2-\frac{1}{2})^2}}$$

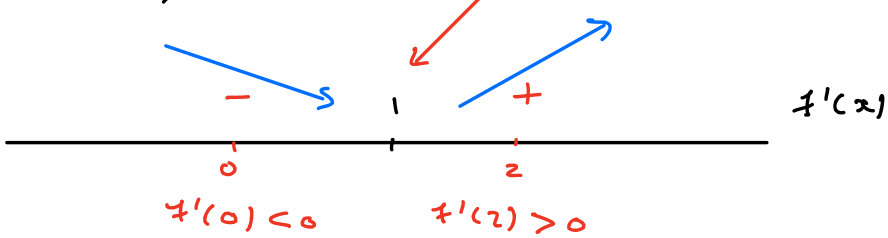
A/ $f'(x) = 0 \iff 4x^3 - 4 = 0 \iff x = 1$

$(2, \frac{1}{2})$ not on $y = x^2$

B/ $\sqrt{(x-2)^2 + (x^2-\frac{1}{2})^2}$ = distance between (x, y) on $y = x^2$ and $(2, \frac{1}{2})$ $\neq 0$

$\Rightarrow$ No B/ points

f cts at 1 $\Rightarrow$ $f(1)$ Abs min



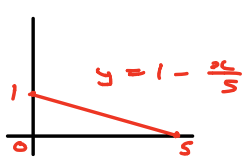
7/ $x = 1 \Rightarrow y = 1^2 = 1 \Rightarrow (1, 1)$ is closest point to $(2, \frac{1}{2})$ on $y = x^2$.

<u>Example</u> Minimize $(x+2)^2 + y^2$ subject to constraint $5y + x = 5$ and $x, y \geq 0$

<u>Objective</u> : Minimize $(x+2)^2 + y^2$

<u>Constraint</u> : $5y + x = 5$ <u>and</u> $x, y \geq 0$

$y = 1 - \frac{x}{5} \Rightarrow (x+2)^2 + (1 - \frac{x}{5})^2 = f(x)$



$\Rightarrow$ Domain $= [0, 5]$

$f'(x) = 2(x+2) - \frac{2}{5}\left(1-\frac{x}{5}\right) = \frac{52}{25}x + \frac{18}{5}$

A/ $f'(x) = 0 \iff x = \frac{\left(-\frac{18}{5}\right)}{\left(\frac{52}{25}\right)} < 0 \implies$ No type A critical numbers in $[0,5]$

B/ Endpoints $0, 5$

$f(0) = 5 \longleftarrow$ minimal value.

$f(5) = 49$

<u>Warning</u>: If we have calculated $f\left(\frac{\left(-\frac{18}{5}\right)}{\left(\frac{52}{25}\right)}\right)$ we would get a smaller value. It's just not in domain. Be very careful to consider correct domain.