New Functions from Old Functions

Translation of Functions

Outside $f(x)$ graph

$c > 0 \implies$

$y = f(x) + c \;=\; y = f(x)$ shifted up by c

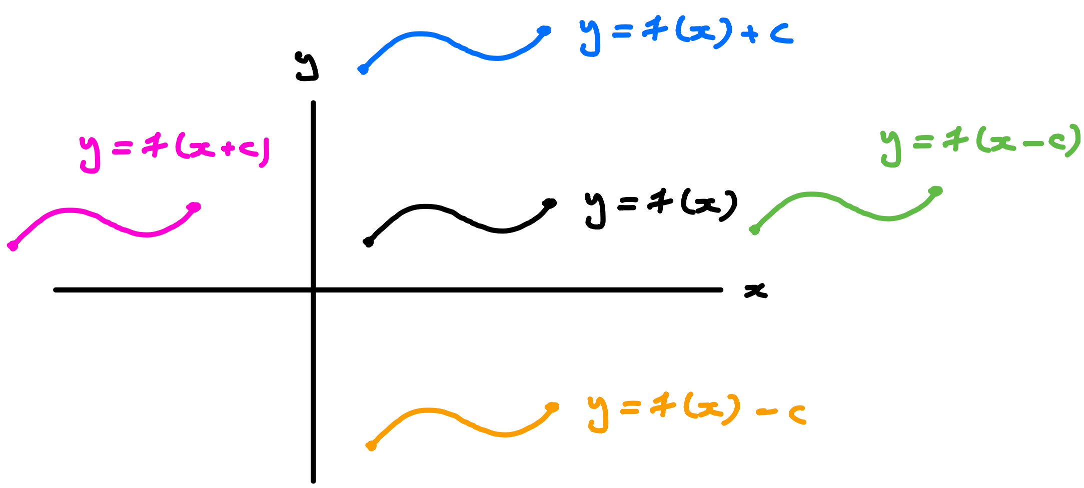
$y = f(x) - c \;=\; y = f(x)$ shifted down by c

Inside $f(x)$

$y = f(x - c) \;=\; y = f(x)$ shifted to right by c

$y = f(x + c) \;=\; y = f(x)$ shifted to left by c

Example



Stretching and Shrinking Functions

$c > 1 \Rightarrow$

outside $f(x)$

$y = cf(x)$ = $y = f(x)$ stretched veritically by c

$y = \frac{1}{c} f(x)$ = $y = f(x)$ shrunk veritically by c

Inside $f(x)$

$y = f(cx)$ = $y = f(x)$ shrunk horizontally by c

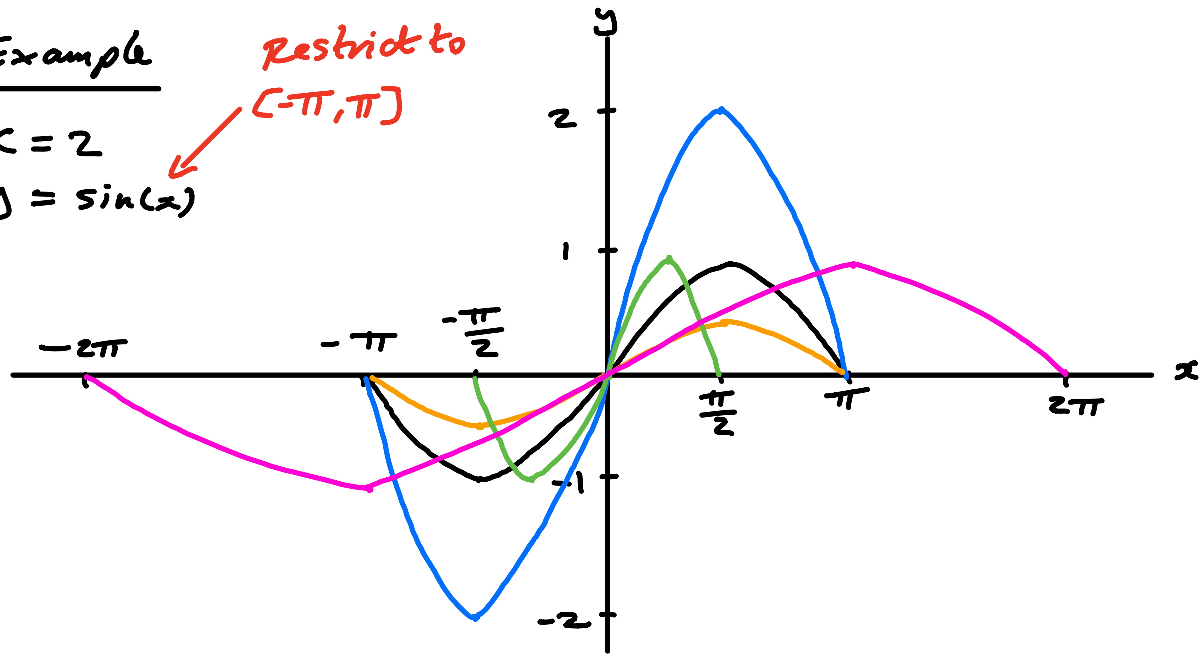
$y = f(\frac{1}{c} x)$ = $y = f(x)$ stretched horizontally by c

Example

$c = 2$

$y = \sin(x)$

Restrict to $[-\pi, \pi]$



Reflecting Functions

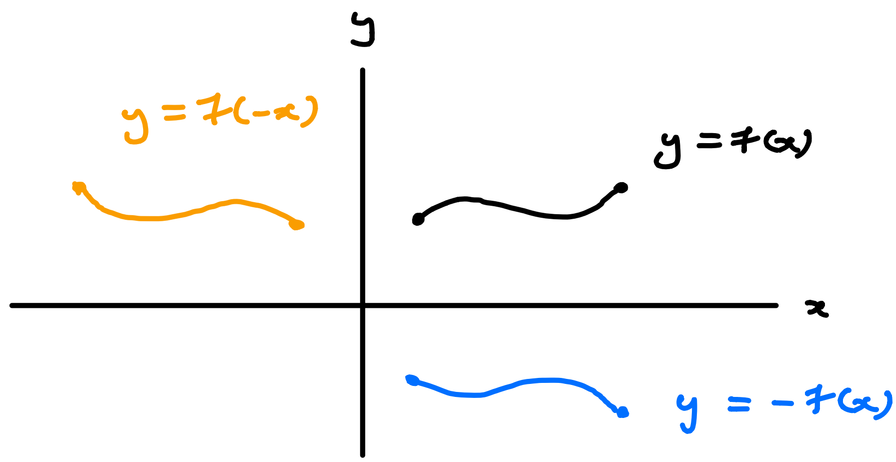
Outside $f(x)$

$y = -f(x)$ = $y = f(x)$ reflected in x-axis

$y = f(-x)$ = $y = f(x)$ reflected in y-axis

Inside $f(x)$

Example



Exercise : Sketch $y = 1 - \cos(2x - \frac{\pi}{2})$ on $[-\pi, \pi]$

Operations on Functions

x must be in domain of f and g

f, g - 2 functions

$f+g$ = function such that $(f+g)(x) = f(x) + g(x)$

fg = function such that $(fg)(x) = f(x)\,g(x)$

f/g = function such that $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$

assuming $g(x) \neq 0$

Example

$f(x) = \sqrt{x}$, $g(x) = \sqrt{2-x}$

$(fg)(x) = \sqrt{x}\sqrt{2-x}$

Need $x \geq 0$ and $2 - x \geq 0$

$\Rightarrow x \geq 0$ and $2 \geq x$

Domain = $[0, 2]$

$$\left(\frac{f}{g}\right)(x) = \frac{\sqrt{x}}{\sqrt{2-x}}$$

Need $x \geq 0$ and

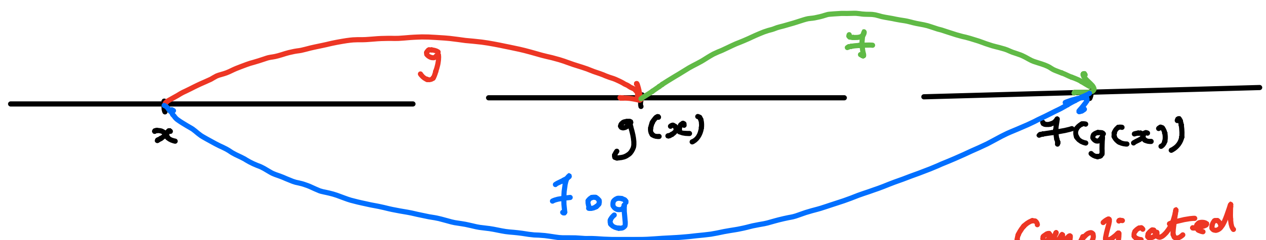
$2 - x > 0$

$\Rightarrow$ $x \geq 0$ and $2 > x$

Domain $= [0, 2)$

Called composition of f and g

$f \circ g$ = function such that $(f \circ g)(x) = f(g(x))$



Complicated set

Domain of $f \circ g$ = x such that

1/ x in domain of g

2/ $g(x)$ in domain of f

in range of g

Examples

1/ $f(x) = \sqrt{x}$, $g(x) = x^2 + 9$

$(f \circ g)(x) = f(g(x)) = \sqrt{x^2 + 9}$

Contained in domain of f

Domain of f = $[0, \infty)$, Range of $f = [0, \infty)$

Domain of g = $\mathbb{R}$, Range of g = $[9, \infty)$

$\Rightarrow$ Domain of $f \circ g$ = $\mathbb{R}$

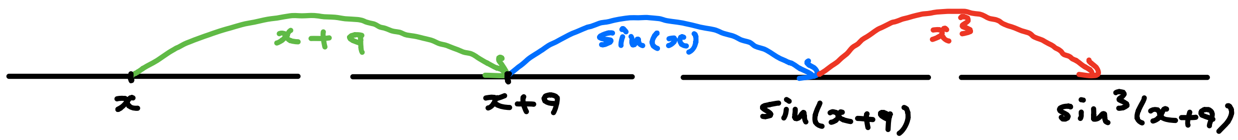
Warning: $x \geq 0$

$(g \circ f)(x) = (\sqrt{x})^2 + 9 = x + 9$

Domain of $g \circ f$ = $[0, \infty)$

Observation $f \circ g \neq g \circ f$ in general.

2/ Express $\sin^3(x+9) = (\sin(x+9))^3$ as a composition of three functions



$h(x) = x+9$, $g(x) = \sin(x)$, $f(x) = x^3$

$\Rightarrow (f \circ g \circ h)(x) = \sin^3(x+9)$

3/ Express $|x|$ as a composition of two algebraic functions.

Let $f(x) = \sqrt{x}$, $g(x) = x^2$

$\Rightarrow (f \circ g)(x) = \sqrt{x^2} = |x|$

positive by definition