# Natural Growth

We say $f$ has <u>natural growth / decay</u> if

$k > 0$ = growth, $k < 0$ = decay

$$f'(t) = k f(t) \quad \text{for all } t.$$

$f'$ is directly proportional to $f$.

## Important Examples

1) Natural Population Growth :

$P(t)$ = population size at time $t$.

$k$ = population growth / decay constant.

2) Radioactive Decay :

$M(t)$ = mass of radioactive material at time $t$.

$k < 0$ and depends on material

$k$ = radioactive decay constant

3) Newtons' Law of Cooling :

$T(t)$ = temperature of object at time $t$ in a room with constant background temperature $T_S$.

$$\Rightarrow \quad \frac{d}{dt}\left(T(t) - T_S\right) = k\left(T(t) - T_S\right)$$

4) Continuous Compound Interest :

$A(t)$ = account balance at time $t$, in savings account with interest rate $r$ ← $0 \leq r \leq 1$

$$\Rightarrow \quad A'(t) = r A(t)$$

For all $t$ $\qquad\qquad$ $C = f(0)$

**Fact:**

$$f'(t) = k f(t) \implies f(t) = Ce^{kt}$$

**Conclusion**: Every above situation is described by a function of form $Ce^{kt}$.

**Examples**

Experiencing natural growth

1/ A population triples in size every 2 years. If the initial population is 1000 what is the population size after 5 years?

Natural Growth $\implies P(t) = 1000\, e^{kt}$.

$$P(t+2) = 3P(t) \implies 1000\, e^{k(t+2)} = 3 \cdot 1000 \cdot e^{kt}$$

$$\implies e^{2k} = 3 \implies k = \frac{\ln(3)}{2}$$

$$\implies P(5) = 1000\, e^{\left(\frac{\ln(3)}{2} \cdot 5\right)}$$

2/ $h$ = Half-Life of radioactive material = Length of time for mass to half.

$$\implies M(t+h) = \frac{1}{2} M(t)$$

$$\implies Ce^{k(t+h)} = \frac{1}{2} Ce^{kt} \implies e^{kh} = \frac{1}{2}$$

$$\implies h = \frac{\ln\left(\frac{1}{2}\right)}{k} = \frac{-\ln(2)}{k}$$

$\Rightarrow$ $$\boxed{\text{Half-Life} = \frac{-\ln(2)}{\text{Decay constant}}}$$

3) $^{12}C$ = Very stable form of Carbon

$^{14}C$ = unstable form of carbon ← $h \simeq 5730$ years

When organic matter dies $^{12}C$ stays same but $^{14}C$ decays. By comparing proportions of $^{14}C$ to $^{12}C$ we can estimate time of death.

E.g. A paper parchment has 80% of $^{14}C$ of living matter. When did the tree die?

$M(t)$ = mass of $^{14}C$, $t$ years after death $= Ce^{kt}$

half-life

$$5730 = \frac{-\ln(2)}{k} \Rightarrow k = \frac{-\ln(2)}{5730}$$

$$\Rightarrow M(t) = Ce^{\frac{-\ln(2)}{5730}t}$$

$$M(t) = 0.8C \Rightarrow Ce^{\frac{-\ln(2)}{5730}t} = 0.8C$$

$$\Rightarrow t = \frac{\ln(0.8)}{\left(\frac{-\ln(2)}{5730}\right)} \approx 1860 \text{ years}$$

Called Carbon Dating

4/ (Discrete to Continuous compound Interest)

Invest \$1 in an account with annual interest rate 100%. ← $r=1$

Discrete Compounding $n$ times in year:

Initial Balance

1 + annual rate

$n=1 \Rightarrow$ Balance at end of year $= 1\cdot(1+1)$

compounding twice at half annual rate

$n=2 \Rightarrow$ Balance at end of year $= 1\cdot(1+\frac{1}{2})^2$

$n=3 \Rightarrow$ Balance at end of year $= 1\cdot(1+\frac{1}{3})^3$

$n \Rightarrow$ Balance at end of year $= 1\cdot(1+\frac{1}{n})^n$

Continuous Compound Interest $= \lim_{n\to\infty}$ Discrete compound Interest

$\Rightarrow$ Balance at end of year after continuously compounding \$1 at 100% interest rate $= \lim_{n\to\infty}(1+\frac{1}{n})^n = e$

<u>General case</u>:

Balance at time $t$ (in years) after continuous compounding at interest rate $r$

Initial balance

Compounding $n$ times per year

$$= \lim_{n\to\infty} C\cdot\left(1+\frac{r}{n}\right)^{nt}$$

$$= C\lim_{n\to\infty}\left(1+\frac{1}{\left(\frac{n}{r}\right)}\right)^{\frac{n}{r}\cdot rt}$$

$$= Ce^{rt}$$ ← As expected