Midterm 2 Review

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \text{slope of tangent line at } (x, y)$$

f - function

$y = f(x) \Rightarrow f'(x) = \frac{d}{dx}(f(x)) = \frac{dy}{dx}$ = Derivative of f with respect to x

Core Derivatives

Power Rule

$$\frac{d}{dx}(x^r) = r x^{r-1}, \quad \frac{d}{dx}(b^x) = \ln(b)\, b^x,$$

$$\frac{d}{dx}(\log_b(x)) = \frac{1}{\ln(b)\, x}, \quad \frac{d}{dx}(\sin(x)) = \cos(x),$$

$$\frac{d}{dx}(\cos(x)) = -\sin(x), \quad \frac{d}{dx}(\tan(x)) = \sec^2(x),$$

$$\frac{d}{dx}(\arcsin(x)) \Big/ \frac{d}{dx}(\arccos(x)) = \frac{1}{\sqrt{1-x^2}} \Big/ \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\arctan(x)) = \frac{1}{1+x^2}$$

Rules of Differentiation

Sum/Difference Rule : $(f \pm g)' = f' \pm g'$

constant

Constant Multiple Rule : $(cf)' = cf'$

Product Rule : $(fg)' = f'g + fg'$

Quotient Rule : $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$

Chain Rule : $\frac{d}{dx}(f(g(x))) = f'(g(x)) \cdot g'(x)$

Important Special Cases :

$$\frac{d}{dx}(e^{g(x)}) = e^{g(x)} \cdot g'(x), \quad \frac{d}{dx}(g(x))^r = r(g(x))^{r-1} \cdot g'(x)$$

$$\frac{d}{dx}(\ln(g(x))) = \frac{g'(x)}{g(x)} \Rightarrow g'(x) = g(x)\frac{d}{dx}(\ln(g(x)))$$

Logarithmic Differentiation.

<u>Worst Case Senario :</u>

$$g(x) = \frac{(x^2+1)^{\tan(x)} \cdot (\ln(x))^x}{\sqrt{(3x+1)}} \Rightarrow g'(x) = ?$$

Log Laws

$$\ln(g(x)) = \tan(x)\ln(x^2+1) + x\ln(\ln(x)) - \frac{1}{2}\ln(3x+1)$$

$$\Rightarrow \frac{d}{dx}\ln(g(x)) = \sec^2(x)\ln(x^2+1) + \tan(x)\frac{2x}{x^2+1} + \ln(\ln(x)) + x\cdot\frac{\left(\frac{1}{x}\right)}{\ln(x)} - \frac{1}{2}\frac{3}{3x+1}$$

$$\Rightarrow g'(x) = \underbrace{\frac{(x^2+1)^{\tan(x)} \cdot (\ln(x))^x}{\sqrt{(3x+1)}}}_{g(x)} \times \underbrace{\left(\sec^2(x)\ln(x^2+1) + \tan(x)\frac{2x}{x^2+1} + \ln(\ln(x)) + x\cdot\frac{\left(\frac{1}{x}\right)}{\ln(x)} - \frac{1}{2}\frac{3}{3x+1}\right)}_{\frac{d}{dx}(\ln(g(x)))}$$

If you see products / quotients of functions like $g(x)^{h(x)}$ use this method.

Overview of Implicit Differentiation

Determining $\frac{dy}{dx}$ from an equation involving x and y

1) Differentiate both sides of equation with respect to x

2) Using laws of differentiation (Product, Chain, Sum ...) expand both sides until everything is expressed in terms of $x, y, \frac{dy}{dx}$

3) Solve in $\frac{dy}{dx}$.

Example Find slope of tangent to $y^3 - x^2 = 2 + y$ at $(2,2)$.

$$\frac{d}{dx}(y^3 - x^2) = \frac{d}{dx}(2+y)$$

Chain Rule

$$\Rightarrow 3y^2 \frac{dy}{dx} - 2x = \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{2x}{3y^2 - 1}$$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{\substack{x=2 \\ y=2}} = \frac{4}{11}$$

Natural Growth / Decay

Constant
$k > 0$ = growth
$k < 0$ = decay

$y = f(t)$ experiences natural growth / decay $\iff \frac{dy}{dt} = ky$

constant $= y(0)$

Fact: $\frac{dy}{dt} = ky \iff y(t) = Ce^{kt}$

Main Examples:

Natural Population Growth, Radioactive Decay. Newton's Law of Cooling.

Example : A radioactive material has mass 10 kg on the 1^{st} of January 2006 and 3 kg on the 1^{st} of January 2015. When will the mass be 2 kg ?

$M(t)$ = Mass at time t (in years after 01/01/2006)

$\Rightarrow M(0) = 10$ and $M(t) = Ce^{kt} \Rightarrow M(t) = 10e^{kt}$

$M(9) = 10e^{9k} = 3 \Rightarrow k = \dfrac{\ln\left(\frac{3}{10}\right)}{9}$

$$\Rightarrow M(t) = 10\, e^{\frac{\ln\left(\frac{3}{10}\right)}{9} t}$$

$$M(t) = 2 \Rightarrow 10\, e^{\frac{\ln\left(\frac{3}{10}\right)}{9} t} = 2$$

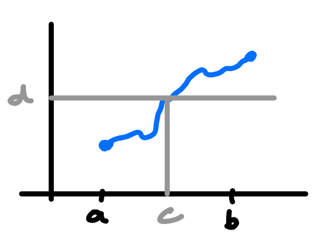
$$\Rightarrow t = \frac{9 \ln\left(\frac{2}{10}\right)}{\ln\left(\frac{3}{10}\right)}$$

The mass is 2kg this many years after 01/01/2006.

Intermediate Value Theorem (I.V.T.)

f – cts on $[a,b]$
$f(a) \leq d \leq f(b)$
or
$f(b) \leq d \leq f(a)$

$\Rightarrow$ There exists c such that $f(c) = d$

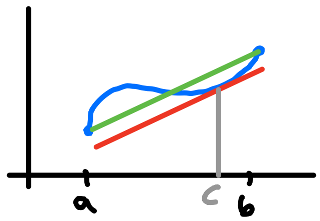


I.V.T. tells us about values of f
Used to prove f takes specific values

Mean Value Theorem (I.V.T.)

f - cts on $[a,b]$
f - diff. on (a,b) $\Rightarrow$ There exists c such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$



M.V.T. tells us about values of f'

Used to prove f' takes specific values

Example : Show that $f(x) = e^x + 2 - 7x^2$ has exactly 3 roots.

differentiable on $\mathbb{R}$

1/ Use I.V.T. to show at least 3 roots.

$f(-1) = e^{-1} + 2 - 7 = e^{-1} - 5 < 0$

$f(0) = 3 > 0$

$f(1) = e + 2 - 7 < 0$

$f(10) = e^{10} + 2 - 700 > 0$

I.V.T. $\Rightarrow$ There are roots in $(-1,0), (0,1), (1,10)$

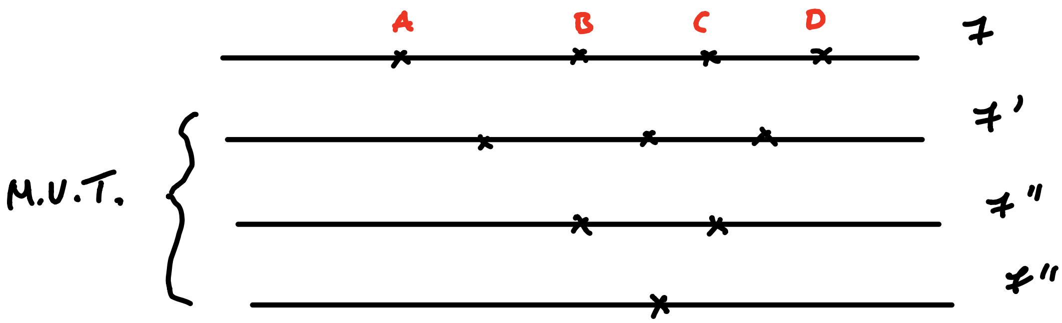
2/ Use M.V.T. to show at most 3 roots.

Assume $f(x)$ has 4 roots A, B, C, D

Observation

$a < b$

M.V.T. $\Rightarrow$ If $g(a) = g(b) = 0 \Rightarrow g'(c) = 0$ for some c in (a,b)



Conclusion

f has 4 roots $\Rightarrow$ f''' has a root

$f(x) = e^x + 2 - 7x^2 \Rightarrow f'''(x) = e^x \neq 0$

$\Rightarrow$ f cannot have 4 roots

$\Rightarrow$ f has exactly 3 roots.

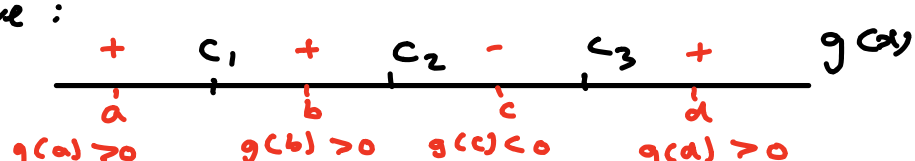
Sign Analysis : Determining when $g(x) > 0$ and $g(x) < 0$.

$g(c) = 0$ $\quad$ g discontinuous at c

1/ Find all type A/ and B/ points.

2/ Draw number line, mark A/ / B/ points and test sign around them.

Picture :



Derivatives and Curve Sketching

Central Facts :

$f'(x) > 0$ on (a, b) $\Rightarrow$ f increasing on (a,b)

$f'(x) < 0$ on (a,b) $\Rightarrow$ f decreasing on (a,b)

$f''(x) > 0$ on (a, b) $\Rightarrow$ f <u>concave up</u> on (a, b)

$f''(x) < 0$ on (a, b) $\Rightarrow$ f <u>concave down</u> on (a, b)

f cts at c

$f'(x)$: $+$ | c | $-$

$f'(x) > 0$ | $f'(x) < 0$

$\Rightarrow$ $f(c)$ local max

f cts at c

$f'(x)$: $-$ | c | $+$

$f'(x) < 0$ | $f'(x) > 0$

$\Rightarrow$ $f(c)$ local min

f cts at c

$f''(x)$: $+$ | c | $-$

$f''(x) > 0$ | $f''(x) < 0$

$\Rightarrow$ $(c, f(c))$ Inflection

f cts at c

$f''(x)$: $-$ | c | $+$

$f''(x) < 0$ | $f''(x) > 0$

$\Rightarrow$ $(c, f(c))$ Inflection

Curve Sketching Checklist:

1/ Domain
2/ Odd/Even
3/ Vertical Asymptotes $\leftarrow$ L'Hospital?
4/ Behavior at $\pm\infty$ $\leftarrow$ Horizontal / Slant asymptotes?
5/ Sign Analysis of f' $\leftarrow$ Increasing / Decreasing / Local Max / Min
6/ Sign Analysis of f'' $\leftarrow$ Concavity / Inflections
7/ Mark key points and put everything together

Finding Absolute Max / Min of $f(x)$

Domain $= [a, b]$
Find critical number and evaluate at f

Domain $= (a, b)$
Pray there is one critical number where there is a local max / min.

Constrained Optimization Checklist

1/ Objective quantity to be maximized / minimized ?

2/ Draw Picture and label unknowns

3/ Give objective in terms of unknowns.

4/ Give constraint in terms of unknowns

5/ Solve constraint in one unknown and sub into objective giving f a single variable function

6/ Identify appropriate domain and find abs. max/min.