MATH 1A MIDTERM 2 (PRACTICE 3)
PROFESSOR PAULIN

DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.

CALCULATORS ARE NOT PERMITTED

YOU MAY USE YOUR OWN BLANK PAPER FOR ROUGH WORK

SO AS NOT TO DISTURB OTHER STUDENTS, EVERYONE MUST STAY UNTIL THE EXAM IS COMPLETE

REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT

THIS EXAM WILL BE ELECTRONICALLY SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED. YOU MAY WRITE SOLUTIONS ON THE BLANK PAGE AT THE BACK BUT BE SURE TO CLEARLY LABEL THEM

Name: ____________________

Student ID: ____________________

GSI's name: ____________________

This exam consists of 5 questions. Answer the questions in the spaces provided.

1. Calculate the following:

(a) (10 points)

$$\frac{d}{dx}(\arcsin(\sqrt{1-x^2}))$$

Solution:

$$\frac{d}{dx}(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}} \qquad \Rightarrow \quad \frac{d}{dx}(\arcsin(\sqrt{1-x^2})) =$$

$$\frac{d}{dx}(1-x^2) = -2x \qquad \frac{1}{\sqrt{1-(\sqrt{1-x^2})^2}} \cdot \frac{1}{2\sqrt{1-x^2}} \cdot -2x$$

(b) (15 points)

$$\lim_{x\to\infty}(x - \ln(2x))$$

Solution:

$$x - \ln(2x) = x\left(1 - \frac{\ln(2x)}{x}\right)$$

$$\lim_{x\to\infty} \frac{\ln(2x)}{x} = \lim_{x\to\infty} \frac{\left(\frac{2}{2x}\right)}{1} = 0 \Rightarrow \lim_{x\to\infty} 1 - \frac{\ln(2x)}{x} = 1$$

$$\lim_{x\to\infty} x = \infty$$

$$\Rightarrow \lim_{x\to\infty} x - \ln(2x) = \infty$$

2. (25 points) An warm object is placed in a room with constant background temperature. It cools according to Newton's Law. At 1pm the object is 40 degrees Celsius, at 2pm the object is 30 degrees Calcius and at 3pm the object is 25 degrees Celcius. What is the temperature of the room?

Solution:

$t = 0$ is 1pm

$T(t)$ = object temperature at time t (in hours)

N.C.C. $\Rightarrow$ $T(t) = Ce^{kt} + T_s$

$T(0) = 40 \Rightarrow C + T_s = 40 \Rightarrow C = 40 - T_s$

$T(1) = 30 \Rightarrow Ce^{k} + T_s = 30 \Rightarrow (40 - T_s)e^{k} + T_s = 30$

$T(2) = 25 \Rightarrow Ce^{2k} + T_s = 25 \Rightarrow (40 - T_s)e^{2k} + T_s = 25$

$$\Rightarrow \quad e^{k} = \frac{30 - T_s}{40 - T_s}, \quad e^{2k} = \frac{25 - T_s}{40 - T_s}$$

$$\Rightarrow \quad \frac{(30 - T_s)^2}{(40 - T_s)^2} = \frac{25 - T_s}{40 - T_s} \Rightarrow (30 - T_s)^2 = (25 - T_s)(40 - T_s)$$

$$\Rightarrow \quad 900 - 60T_s + T_s^2 = 1000 - 65T_s + T_s^2$$

$$\Rightarrow \quad 5T_s = 100 \Rightarrow T_s = 20^\circ C.$$

3. (25 points) Sketch the following curve. Be sure to indicate asymptotes, local maxima and minima and concavity. Show your working on this page and draw the graph on the next page.

$$y = x^{2/3} - x^{5/3}$$

You do not need to give exactly y-coordinates for inflections and local extrema.

Solution:

$$f(x) = x^{2/3} - x^{5/3} = x^{2/3}(1-x)$$

Domain : $\mathbb{R}$

Odd/Even : Neither

Vertical asymptotes : None

Behavior at $\pm\infty$

$$\lim_{x\to\pm\infty} x^{2/3} = \infty$$

$$\lim_{x\to\infty} 1-x = -\infty$$

$$\lim_{x\to-\infty} 1-x = \infty$$

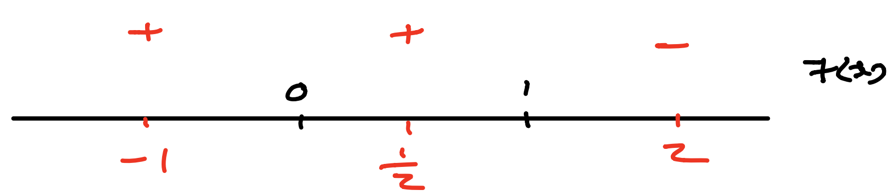
$$\Rightarrow \quad \lim_{x\to\infty} x^{2/3}(1-x) = -\infty$$

$$\lim_{x\to-\infty} x^{2/3}(1-x) = \infty$$

Sign Analysis for f

A/ $x = 0, 1$

B/ None

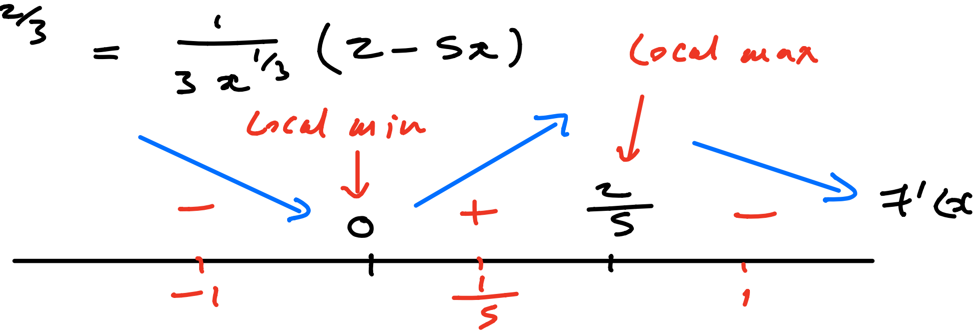


$$f'(x) = \frac{2}{3}x^{-\frac{1}{3}} - \frac{5}{3}x^{2/3} = \frac{1}{3x^{1/3}}(2-5x)$$

Sign Analysis for f'

A/ $x = \frac{2}{5}$

B/ $x = 0$



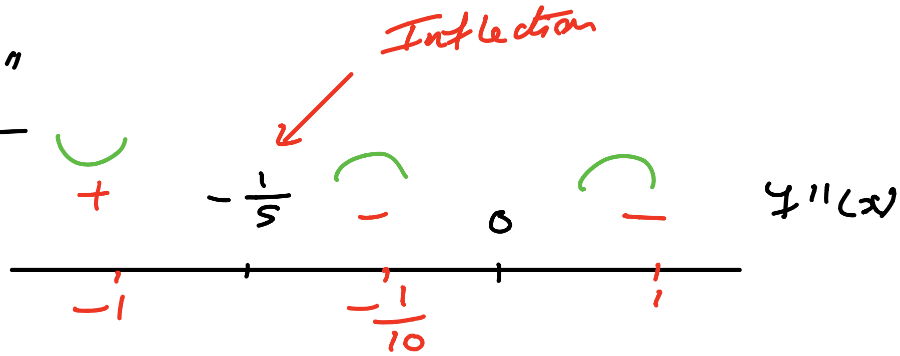
Solution (continued) :

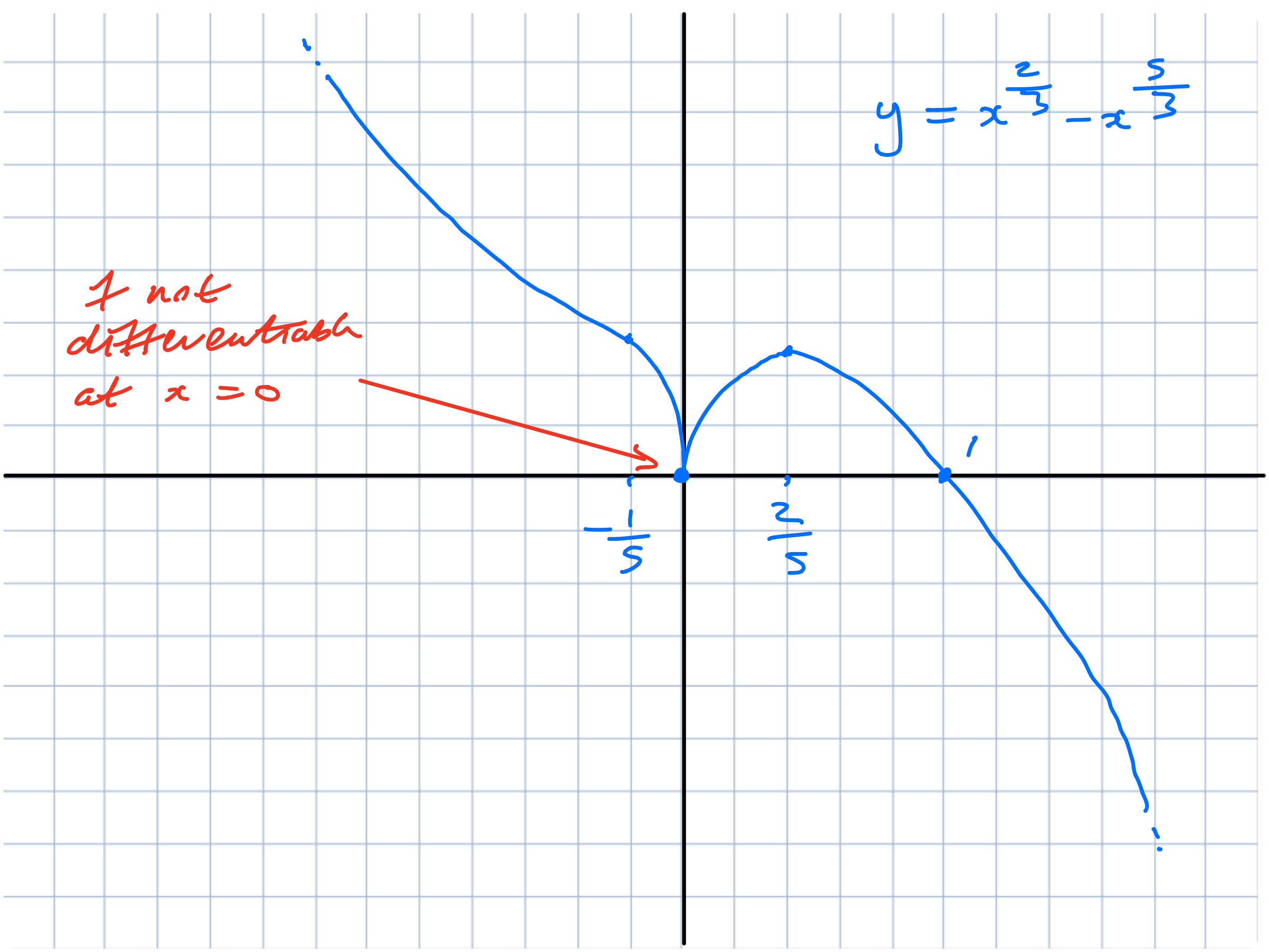
$$f''(x) = \frac{-2}{9} x^{-\frac{4}{3}} - \frac{10}{9} x^{-\frac{1}{3}} = \frac{-2}{9} x^{-\frac{4}{3}} (1 + 5x)$$

Sign Analysis for f''

A/ $x = \frac{-1}{5}$

B/ $x = 0$





4. (25 points) Let $f(x) = \sqrt{x}$. What are the absolute extrema of the derivative, $f'(x)$, on the interval $[4, 5]$? Using this information, along with the Mean Value Theorem, prove that

$$20/9 \leq \sqrt{5} \leq 9/4$$

Solution:

$$f'(x) = \frac{1}{2\sqrt{x}} = \frac{1}{2} x^{-\frac{1}{2}} \Rightarrow f''(x) = \frac{-1}{4} \cdot x^{-\frac{3}{2}} = \frac{-1}{4x^{3/2}}$$

A/ $f''(x) = 0$ has no solutions

B/ $f''(x)$ exists for $4 < x < 5$ $\Rightarrow$ Only endpoints $4, 5$

$$f'(4) = \frac{1}{4} > f'(5) = \frac{1}{2\sqrt{5}} \Rightarrow \frac{1}{2\sqrt{5}} \leq f'(x) \leq \frac{1}{4}$$

for all x in $[4, 5]$

Abs. Max

Abs. Min

$$\text{M.V.T} \Rightarrow \frac{1}{2\sqrt{5}} \leq \frac{f(5) - f(4)}{5-1} \leq \frac{1}{4}$$

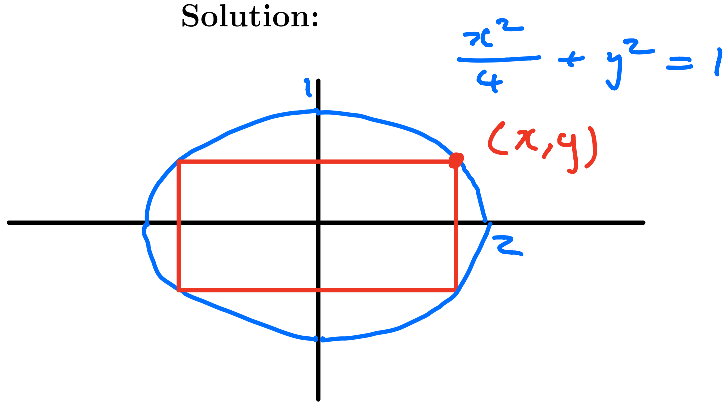
$$\Rightarrow \frac{1}{2\sqrt{5}} \leq \sqrt{5} - 2 \leq \frac{1}{4} \Rightarrow 1 \leq 10 - 4\sqrt{5} \leq \frac{1}{2}\sqrt{5}$$

$$1 \leq 10 - 4\sqrt{5} \Rightarrow \sqrt{5} \leq \frac{9}{4}$$

$$10 - 4\sqrt{5} \leq \frac{1}{2}\sqrt{5} \Rightarrow 10 \leq \frac{9}{2}\sqrt{5} \Rightarrow \frac{20}{9} \leq \sqrt{5}$$

5. (25 points) Find the area of the largest rectangle that can be inscribed in the ellipse $x^2/4 + y^2 = 1$.

Solution:



Q = Area of rectangle

$= 2x \cdot 2y = 4xy$

Constraint :

$\frac{x^2}{4} + y^2 = 1$ and $0 \le x \le 2$

$$\Rightarrow \quad y = \frac{1}{2}\sqrt{4-x^2}$$

$0 \le x \le 2$

$$\Rightarrow \quad Q = 4xy = 2x\sqrt{4-x^2} = f(x)$$

$$f'(x) = 2\sqrt{4-x^2} + 2x \cdot \frac{-2x}{2\sqrt{4-x^2}} = \frac{8-4x^2}{\sqrt{4-x^2}}$$

A/ $f'(x) = 0 \iff x = \sqrt{2}$ in $[0,2]$

B/ f' undefined $\iff x = 0, 2$

$$f(0) = 0, \quad f(2) = 0, \quad f(\sqrt{2}) = 2 \cdot \sqrt{2}\left(\sqrt{4-(\sqrt{2})^2}\right) = 4 > 0$$

Abs. Max on $[0,2]$

$\Rightarrow$ Largest Area.

END OF EXAM