MATH 1A MIDTERM 2 (PRACTICE 2)
PROFESSOR PAULIN

DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.
CALCULATORS ARE NOT PERMITTED
YOU MAY USE YOUR OWN BLANK PAPER FOR ROUGH WORK
SO AS NOT TO DISTURB OTHER STUDENTS, EVERYONE MUST STAY UNTIL THE EXAM IS COMPLETE
REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT
THIS EXAM WILL BE ELECTRONICALLY SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED. YOU MAY WRITE SOLUTIONS ON THE BLANK PAGE AT THE BACK BUT BE SURE TO CLEARLY LABEL THEM

Name: ________________________________

Student ID: ________________________________

GSI's name: ________________________________

This exam consists of 5 questions. Answer the questions in the spaces provided.

1. Calculate the following:

   (a) (10 points)

$$\frac{d}{dx}\sqrt{\arctan(x^3)+1}$$

Solution:

$$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx}\arctan(x) = \frac{1}{x^2+1} \Rightarrow \frac{d}{dx}\sqrt{\arctan(x^3)+1} = \frac{1}{2\sqrt{\arctan(x^3)+1}} \cdot \frac{1}{(x^3)^2+1} \cdot 3x^2$$

$$\frac{d}{dx}(x^3) = 3x^2$$

   (b) (15 points)

$$\lim_{x\to\infty} x^{1/x}$$

Solution:

$$x^{\frac{1}{x}} = e^{\frac{\ln(x)}{x}}$$

L'Hospital

$$\lim_{x\to\infty} \frac{\ln(x)}{x} = \lim_{x\to\infty} \frac{\left(\frac{1}{x}\right)}{1} = 0$$

$$\Rightarrow \lim_{x\to\infty} x^{\frac{1}{x}} = \lim_{x\to\infty} e^{\frac{\ln(x)}{x}} = e^{\lim_{x\to\infty}\frac{\ln(x)}{x}} = e^0 = 1$$

2. (25 points) Consider the curve given by the equation $4x^2+y^2=4$. Determine all tangent lines to this curve which pass through the point $(2,0)$.

Solution:

$$\frac{d}{dx}(4x^2+y^2) = \frac{d}{dx}(4) \Rightarrow 8x + 2y\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-4x}{y}$$

Slope of tangent at (x,y)

Slope of line through (x,y) and $(2,0)$

$$\frac{-4x}{y} = \frac{y-0}{x-2} \Rightarrow -4x(x-2) = y^2$$

$$4x^2+y^2=4$$

$$\Rightarrow \quad y^2 = 4-4x^2$$

$$-4x(x-2) = y^2$$

$$\Rightarrow \quad -4x(x-2) = 4-4x^2$$

$$\Rightarrow \quad -4x^2+8x = 4-4x^2$$

$$\Rightarrow \quad 8x = 4 \Rightarrow x = \frac{1}{2}$$

$$x=\frac{1}{2} \Rightarrow 4\cdot\left(\frac{1}{2}\right)^2 + y^2 = 4 \Rightarrow y^2 = 3 \Rightarrow y = \pm\sqrt{3}$$

There are two tangent lines that pass through $(2,0)$, the one at $(\frac{1}{2}, \sqrt{3})$ and the one at $(\frac{1}{2}, -\sqrt{3})$.

3. (25 points) Sketch the following curve. Be sure to indicate asymptotes, local maxima and minima and concavity. Show your working on this page and draw the graph on the next page.

$$y = \frac{e^{2x}}{x}$$

Solution:

Domain : $(-\infty, 0) \cup (0, \infty)$

Odd / Even : Neither

Vertical Asymptotes :

$$\lim_{x \to 0} e^{2x} = 1 > 0$$

$$\lim_{x \to 0^+} x = 0^+ , \quad \lim_{x \to 0^-} x = 0^- \quad \Rightarrow \quad \lim_{x \to 0^+} \frac{e^{2x}}{x} = \infty , \quad \lim_{x \to 0^-} \frac{e^{2x}}{x} = -\infty$$

Behavior at $\pm\infty$

L'Hospital

$$\lim_{x \to \infty} \frac{e^{2x}}{x} = \lim_{x \to \infty} \frac{2e^{2x}}{1} = \infty$$

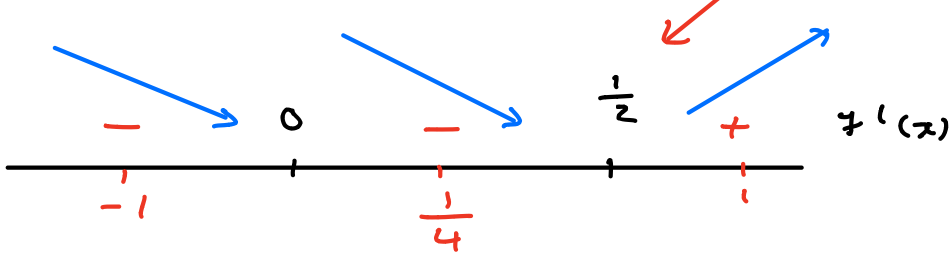
$$\lim_{x \to -\infty} e^{2x} = 0 \Rightarrow \lim_{x \to -\infty} \frac{e^{2x}}{x} = 0 \quad \Rightarrow \quad y = 0 \text{ horizontal asymptote}$$

$$f'(x) = \frac{2e^{2x}x - e^{2x}}{x^2} = e^{2x}\left(\frac{2x-1}{x^2}\right) = e^{2x}\left(\frac{2}{x} - \frac{1}{x^2}\right)$$

$f(\frac{1}{2}) = 2e$ local min

A/ $x = \frac{1}{2}$

B/ $x = 0$



Solution (continued) :

$$f''(x) = 2e^{2x}\left(\frac{2}{x} - \frac{1}{x^2}\right) + e^{2x}\left(\frac{-2}{x^2} + \frac{2}{x^3}\right)$$

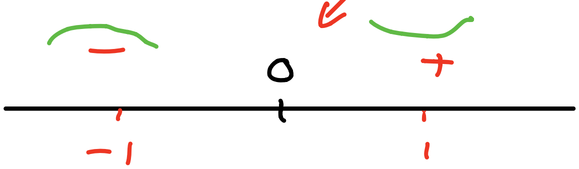
$$= e^{2x}\left(\frac{4x^2-2x-2x+2}{x^3}\right) = 2e^{2x}\left(\frac{2x^2-2x+1}{x^3}\right)$$

$2x^2-2x+1 \neq 0$ for all x as $(-2)^2 - 4\cdot 2\cdot 1 < 0$

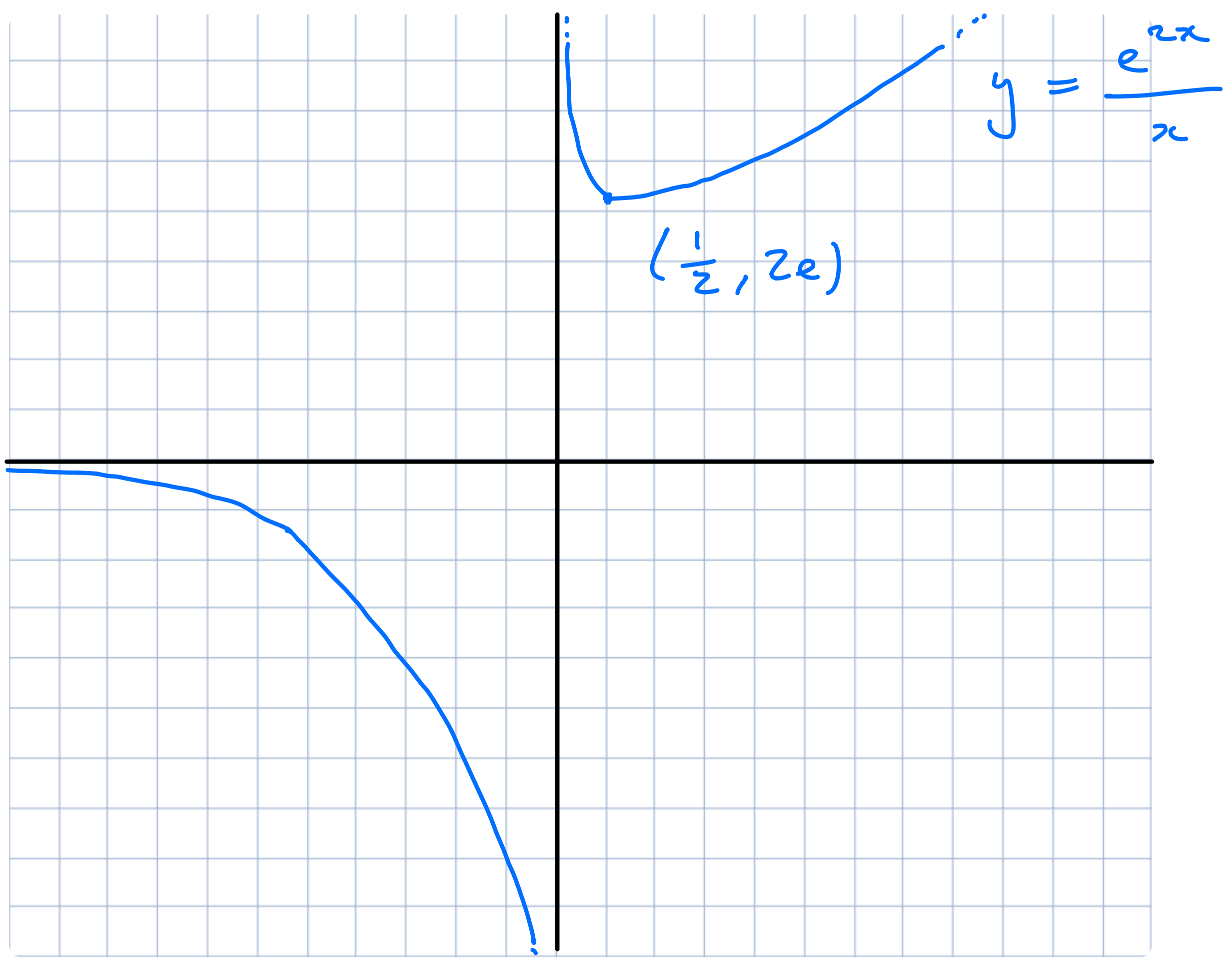
A/ None

B/ $x = 0$

Not Inflection



$f''(x)$



4. (25 points) Show that the following equation has exactly one real solution. Be sure to carefully justify you answer clearly stating any results you use from lectures.

$$\arctan(x) = 2 + 7x$$

Solution:

Continuous on $\mathbb{R}$

Let $f(x) = \arctan(x) - 2 - 7x$

$-\frac{\pi}{2} < \arctan(x) < \frac{\pi}{2} \Rightarrow f(10) < 0$, $f(-10) > 0$

I.V.T. $\Rightarrow$ There exists x such that $f(x) = 0$

Assume there exist $a < b$ such that $f(a) = f(b) = 0$

M.V.T. $\Rightarrow$ There exists c such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} = 0$$

$$f'(x) = \frac{1}{x^2+1} - 7 < 0 \quad \text{for all } x$$

Hence no such c exists $\Rightarrow$ There is exactly one solution to $f(x) = 0$

5. (25 points) What is the maximum possible value of $x+6y$ subject to the constraint $x+y^2=4$, where x and y are non-negative real numbers.

Solution:

Objective : Maximize $x+6y$

Constraint : $x+y^2=4$, $x, y \geq 0$

$x+y^2=4 \Rightarrow x=4-y^2 \Rightarrow x+6y = (4-y^2)+6y = f(y)$

$x, y \geq 0$ and $x+y^2=4 \Rightarrow y \leq 2$

Domain $= [0,2]$

$f'(y) = -2y+6$

A/ $f'(y)=0 \Rightarrow y=3$

B/ f' continuous everywhere

$\Rightarrow$ 0, 2 are only critical numbers on $[0,2]$

$f(0) = 4$

$f(2) = 12$

$\Rightarrow$ The maximum possible value of $x+6y$ is 12 under the condition $x+y^2=4$ and $x, y \geq 0$

END OF EXAM