MATH 1A MIDTERM 2 (PRACTICE 1)
PROFESSOR PAULIN

DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.

CALCULATORS ARE NOT PERMITTED

YOU MAY USE YOUR OWN BLANK PAPER FOR ROUGH WORK

SO AS NOT TO DISTURB OTHER STUDENTS, EVERYONE MUST STAY UNTIL THE EXAM IS COMPLETE

REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT

THIS EXAM WILL BE ELECTRONICALLY SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED. YOU MAY WRITE SOLUTIONS ON THE BLANK PAGE AT THE BACK BUT BE SURE TO CLEARLY LABEL THEM

Name: ______________________________

Student ID: ______________________________

GSI's name: ______________________________

This exam consists of 5 questions. Answer the questions in the spaces provided.

1. Determine the derivatives of the following functions (you do not need to use the limit definition):

   (a) (10 points)

$$e^{\frac{\arccos(x)}{x}}$$

Solution:

$$\frac{d}{dx}\left(\frac{\arccos(x)}{x}\right) = \frac{\frac{-1}{\sqrt{1-x^2}}\cdot x - 1\cdot\arccos(x)}{x^2}$$

Chain Rule $\Rightarrow$

$$\frac{d}{dx}\left(e^{\left(\frac{\arccos(x)}{x}\right)}\right) = e^{\frac{\arccos(x)}{x}}\cdot\left(\frac{\frac{-1}{\sqrt{1-x^2}}\cdot x - 1\cdot\arccos(x)}{x^2}\right)$$

   (b) (15 points)

$$\sqrt{x}^{\sqrt{x}}$$

Solution:

$$f(x) = \sqrt{x}^{\sqrt{x}} \Rightarrow \ln(f(x)) = \sqrt{x}\ln(\sqrt{x}) = \frac{1}{2}\sqrt{x}\ln(x)$$

$$\frac{d}{dx}\ln(f(x)) = \frac{1}{4\sqrt{x}}\ln(x) + \frac{1}{2}\cdot\sqrt{x}\cdot\frac{1}{x}$$

$$f'(x) = f(x)\frac{d}{dx}\ln(f(x)) = \sqrt{x}^{\sqrt{x}}\left(\frac{1}{4\sqrt{x}}\ln(x) + \frac{1}{2}\cdot\sqrt{x}\cdot\frac{1}{x}\right)$$

2. (25 points) Find the equations of the tangent and normal lines to the following curve at the given point.

$$x^2 + 4xy = 13 - y^2, \qquad (2, 1).$$

Solution:

$$\frac{d}{dx}(x^2 + 4xy) = \frac{d}{dx}(13 - y^2)$$

$$\Rightarrow \quad 2x + \frac{d}{dx}(4xy) = \frac{d}{dx}(-y^2)$$

$$\Rightarrow \quad 2x + 4y + 4x\frac{dy}{dx} = -2y\frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{-2x - 4y}{4x + 2y}$$

$$\left.\frac{dy}{dx}\right|_{\substack{x=2 \\ y=1}} = \frac{-2\cdot 2 - 4\cdot 1}{4\cdot 2 + 2\cdot 1} = \frac{-8}{10}$$

$\Rightarrow$ Tangent at $(2,1)$ has equation $\quad y - 1 = \frac{-8}{10}(x-2)$

Normal at $(2,1)$ has equation $\quad y - 1 = \frac{10}{8}(x-2)$

3. (25 points) Show that the following equation has at most 2 real solutions. Be sure to carefully justify you answer clearly stating any results you use from lectures.

$$3x^6 + 4x^2 + c = 0, \quad \text{where } c \text{ is any constant}$$

Solution:

Assume $f(x) = 3x^6 + 4x^2 + c$ has 3 real roots $a < b < c$

$\Rightarrow f(a) = f(b) = f(c)$.

M.V.T. $\Rightarrow$ There exists $a < d < b$ such that $f'(d) = \dfrac{f(b) - f(a)}{b - a} = 0$

There exists $b < e < c$ such that $f'(e) = \dfrac{f(c) - f(b)}{c - b} = 0$

M.V.T. $\Rightarrow$ There exists $d < g < e$ such that $f''(g) = \dfrac{f'(e) - f'(d)}{e - d} = 0$

Conclusion :

$f(x) = 0$ has 3 real solutions $\Rightarrow$ $f''(x) = 0$ has at least one real solution.

$f'(x) = 18x^5 + 8x \Rightarrow f''(x) = 90x^4 + 8$

$\Rightarrow f''(x) \geq 8 > 0$ for all x.

Hence $f''(x) = 0$ has no real solutions, so $f(x) = 0$ has at most 2 solutions.

4. (25 points) Sketch the following curve. Be sure to indicate asymptotes, local maxima and minima and concavity. Show your working on this page and draw the graph on the next page.

$$y = \frac{x^3}{(x+1)^2}$$

Solution:

Domain : $(-\infty, -1) \cup (-1, \infty)$

Odd / Even : Neither

Vertical Asymptotes

$\lim_{x \to -1} x^3 = -1 < 0$

squares are positive

$\lim_{x \to -1} (x+1)^2 = 0^+$

$\Rightarrow \lim_{x \to -1} \frac{x^3}{(x+1)^2} = -\infty$

$\Rightarrow x = -1$ vertical asymptote

Behavior at $\pm\infty$

$$\lim_{x \to \pm\infty} \frac{f(x)}{x} = \lim_{x \to \pm\infty} \frac{x^3}{x^3 + 2x^2 + x} = 1$$

$$\lim_{x \to \pm\infty} f(x) - x = \lim_{x \to \pm\infty} \frac{x^3}{(x+1)^2} - \frac{x(x+1)^2}{(x+1)^2} = \lim_{x \to \pm\infty} \frac{-2x^2 - x}{x^2 + 2x + 1} = -2$$

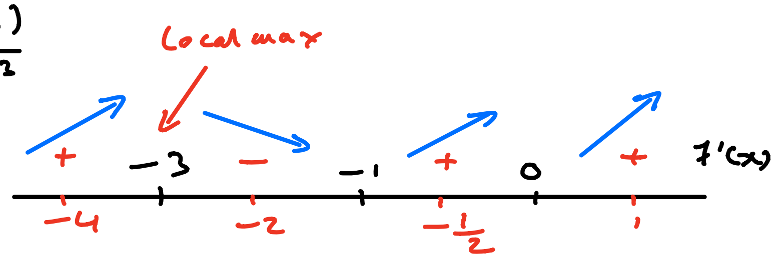
$\Rightarrow y = x - 2$ slant asymptote.

$$f'(x) = \frac{3x^2(x+1)^2 - 2(x+1)x^3}{(x+1)^4} = \frac{3x^2(x+1) - 2x^3}{(x+1)^3}$$

$$= \frac{x^3 + 3x^2}{(x+1)^3} = \frac{x^2(x+3)}{(x+1)^3}$$

A/ $f'(x) = 0 \Leftrightarrow x = 0, -3$

B/ f' discontinuous $\Leftrightarrow x = -1$



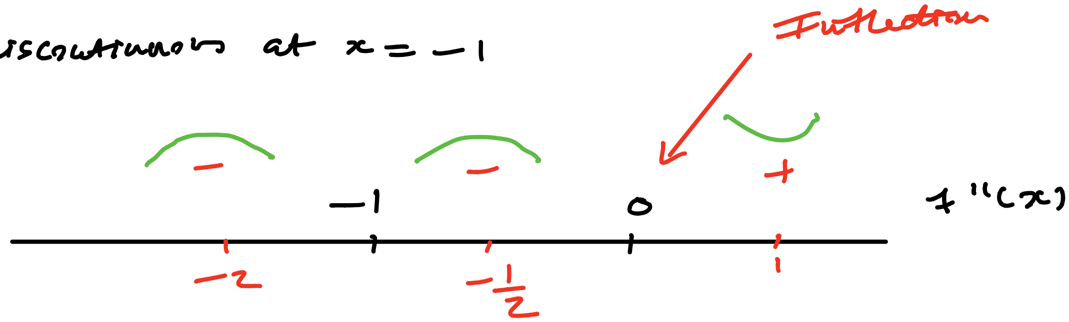
Solution (continued) :

$$f''(x) = \frac{(3x^2+6x)(x+1)^3 - 3(x+1)^2(x^3+3x^2)}{(x+1)^6}$$

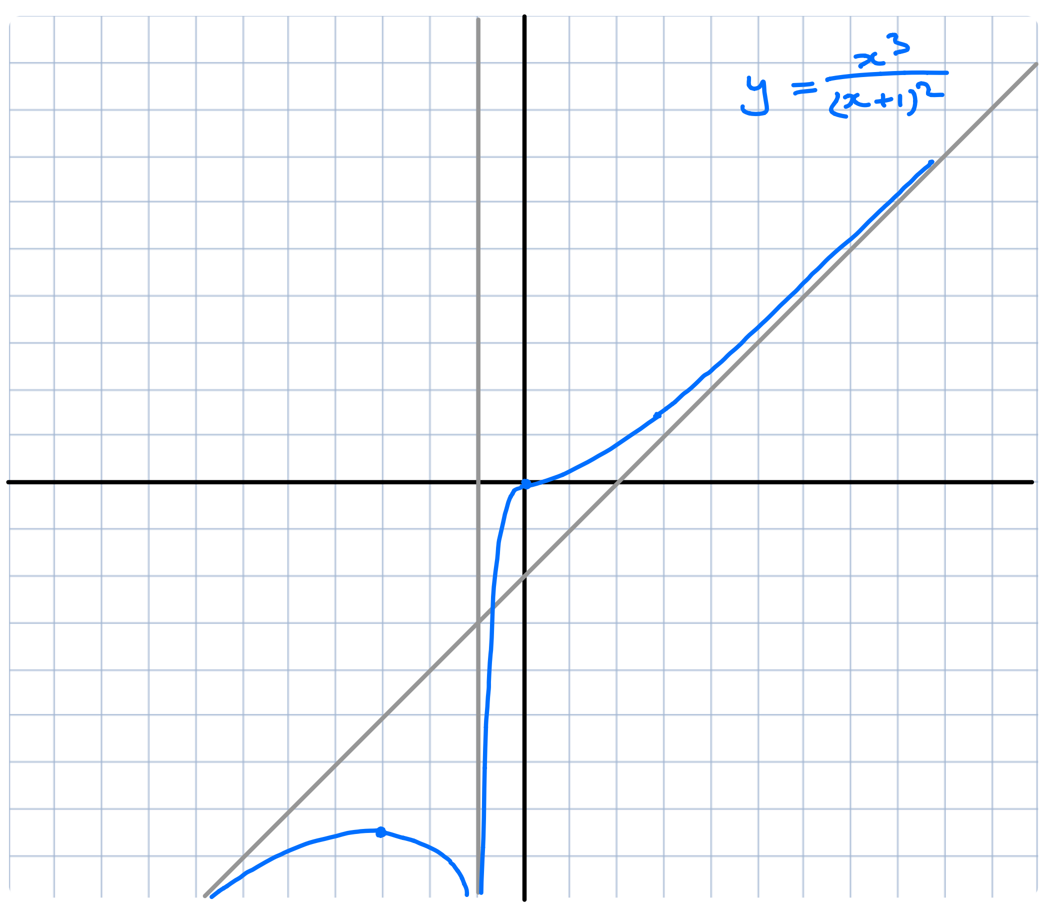
$$= \frac{(3x^2+6x)(x+1) - 3(x^3+3x^2)}{(x+1)^4} = \frac{6x}{(x+1)^4}$$

A/ $f''(x) = 0 \iff x = 0$

B/ f'' discontinuous at $x = -1$



$f(0) = 0$, $f(-3) = \frac{-27}{4}$



5. (25 points) Find the point on the curve $y^2 + 9x^2 = 36$ which is closest to (1,0). Hint: When minimizing the objective function make sure you think carefully about the domain.

Solution:

$$Q = \sqrt{(x-1)^2 + y^2}$$

Constraint : $y^2 + 9x^2 = 36 \Rightarrow y^2 = 36 - 9x^2$
$= 9(4 - x^2)$

$\Rightarrow -2 \le x \le 2$

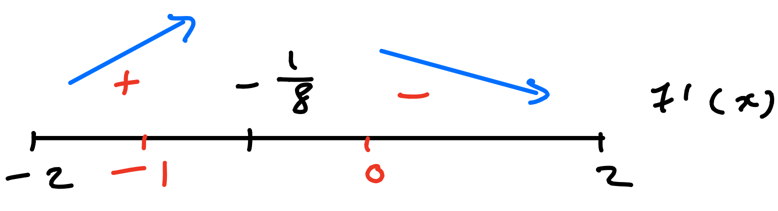
and $Q = \sqrt{(x-1)^2 + 36 - 9x^2} = f(x)$

$$f'(x) = \frac{2(x-1) - 18x}{2\sqrt{(x-1)^2 + 36 - 9x^2}} = \frac{-16x - 2}{2\sqrt{(x-1)^2 + 36 - 9x^2}}$$

A/ $f'(x) = 0 \Leftrightarrow x = \frac{-1}{8}$

B/ f' continuous on $[-2, 2]$.

Denominator is non-zero on $[-2, 2]$



$f'(-1) > 0$ $f'(0) < 0$

$\Rightarrow$ Absolute min is at -2 or 2

$f(2) = 1$
$f(-2) = 3$
$\Rightarrow$ $(2, 0)$ is closest point

END OF EXAM