MATH 1A MIDTERM 2 (002)
PROFESSOR PAULIN

DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.

CALCULATORS ARE NOT PERMITTED

YOU MAY USE YOUR OWN BLANK PAPER FOR ROUGH WORK

SO AS NOT TO DISTURB OTHER STUDENTS, EVERYONE MUST STAY UNTIL THE EXAM IS COMPLETE

REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT

THIS EXAM WILL BE ELECTRONICALLY SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED. YOU MAY WRITE SOLUTIONS ON THE BLANK PAGE AT THE BACK BUT BE SURE TO CLEARLY LABEL THEM

Name: ____________________

Student ID: ____________________

GSI's name: ____________________

This exam consists of 5 questions. Answer the questions in the spaces provided.

1. Determine the derivatives of the following functions (you do not need to use the limit definition and you do not need to simplify your answer):

(a) (10 points)

$$\frac{\arctan(x^2)}{x}$$

Solution:

$$\frac{d}{dx}\left(\frac{\arctan(x^2)}{x}\right) = \frac{\frac{d}{dx}(\arctan(x^2))x - \arctan(x^2)\cdot\frac{d}{dx}(x)}{x^2}$$

$$= \frac{\frac{1}{1+(x^2)^2}\cdot 2x^2 - \arctan(x^2)}{x^2}$$

(b) (15 points)

$$(\sin x)^{\cos x}$$

Solution:

$$f(x) = (\sin(x))^{\cos(x)} \Rightarrow \frac{d}{dx}\ln(f(x)) = \frac{d}{dx}(\cos(x)\ln(\sin(x)))$$

$$= -\sin(x)\ln(\sin(x)) + \cos(x)\frac{\cos(x)}{\sin(x)}$$

$$\Rightarrow \frac{d}{dx}(\sin(x)^{\cos(x)}) = \sin(x)^{\cos(x)}\left(-\sin(x)\ln(\sin(x)) + \cos(x)\frac{\cos(x)}{\sin(x)}\right)$$

2. (25 points) A warm object is placed in a cool room. The room's temperature is 10^oC. At 2pm the object has temperature 30^oC. At 3pm the temperature is 15^oC. At what time was the temperature of the object 50^oC?

Solution:

in hours

2pm : $t = 0$

$$T(t) = T_s + Ce^{kt}$$

$$\left.\begin{array}{l} T_s = 10 \\ T(0) = 30 \\ T(1) = 15 \end{array}\right\} \Rightarrow \begin{array}{l} 10 + C = 30 \\ 10 + Ce^k = 15 \end{array} \Rightarrow \begin{array}{l} C = 20 \\ e^k = \frac{5}{20} \end{array} \Rightarrow k = \ln\left(\frac{1}{4}\right)$$

$$\Rightarrow T(t) = 10 + 20e^{\ln(\frac{1}{4})t} = 10 + 20\cdot\left(\frac{1}{4}\right)^t$$

$$T(t) = 50 \Rightarrow 10 + 20\left(\frac{1}{4}\right)^t = 50 \Rightarrow \left(\frac{1}{4}\right)^t = 2$$

$$\Rightarrow t\ln(2^{-2}) = \ln(2) \Rightarrow t = \frac{\ln(2)}{-2\ln(2)} = \frac{-1}{2}$$

$\Rightarrow$ Temperature was $50^\circ C$ at 1.30 pm.

3. (25 points) Let f be a function which is differentiable on $\mathbb{R}$. Assume that

$$2 < f'(x) < 5 \text{ for all } x \text{ in } \mathbb{R}$$

If $f(4) = 2$, what are the possible values of $f(1)$?

Solution:

M.V.T. $\Rightarrow$ $\dfrac{f(4) - f(1)}{4-1} = f'(c)$ for some c in $(1, 4)$

$2 < f'(c) < 5 \Rightarrow 2 < \dfrac{f(4) - f(1)}{4-1} < 5$

$\Rightarrow \quad 2 < \dfrac{2 - f(1)}{3} < 5$

$\Rightarrow \quad 6 < 2 - f(1) < 15$

$\Rightarrow \quad 4 < -f(1) < 13 \Rightarrow -13 < f(1) < -4$

$\Rightarrow$ $f(1)$ can take any value in $(-13, -4)$

4. (25 points) Sketch the following curve. Be sure to indicate asymptotes, local maxima and minima and concavity. Show your working on this page and draw the graph on the next page.

$$y = \frac{-x^2 + 5x - 7}{x - 2} = f(x)$$

Solution:

$$\frac{-9+15-7}{1} = -1$$

$$-1+5-7 = -3, \quad \frac{-3}{-1} = 3$$

Domain: $(-\infty, 2) \cup (2, \infty)$

Odd/Even: Neither

Vertical Asymptotes

$$\left.\begin{aligned} &\lim_{x \to 2} -x^2 + 5x - 7 = -1 < 0 \\ &\lim_{x \to 2^-} x - 2 = 0^-, \quad \lim_{x \to 2^+} x - 2 = 0^+ \end{aligned}\right\} \Rightarrow \begin{aligned} &\lim_{x \to 2^-} f(x) = \infty \\ &\lim_{x \to 2^+} f(x) = -\infty \end{aligned}$$

Behavior at $\pm\infty$

$$\lim_{x \to \pm\infty} \frac{f(x)}{x} = \lim_{x \to \pm\infty} \frac{-x^2 + 5x - 7}{x^2 - 2x} = \frac{-1}{1} = -1 \text{ "} m$$

$$\lim_{x \to \pm\infty} f(x) - mx = \lim_{x \to \pm\infty} \frac{\cancel{-x^2} + 5x - 7 \cancel{+x^2} - 2x}{x - 2} = 3 \text{ = } b$$

$$f(x) = \frac{-x^2 + 5x - 7}{x - 2}$$

A/ None $(-x^2 + 5x - 7 \neq 0)$

B/ $x = 2$

V.A.

$+$ 2 $-$ f

$f(0) > 0$ $f(3) < 0$

$$f'(x) = \frac{(-2x+5)(x-2) - (-x^2+5x-7)}{(x-2)^2} = \frac{-x^2 + 4x - 3}{(x-2)^2}$$

$$= -\frac{(x-3)(x-1)}{(x-2)^2}$$

Solution (continued) :

A/ $x = 1, 3$

B/ $x = 2$

local min, V.A., local max

f': $-$ on $(-\infty,1)$, $+$ on $(1,2)$, $+$ on $(2,3)$, $-$ on $(3,\infty)$

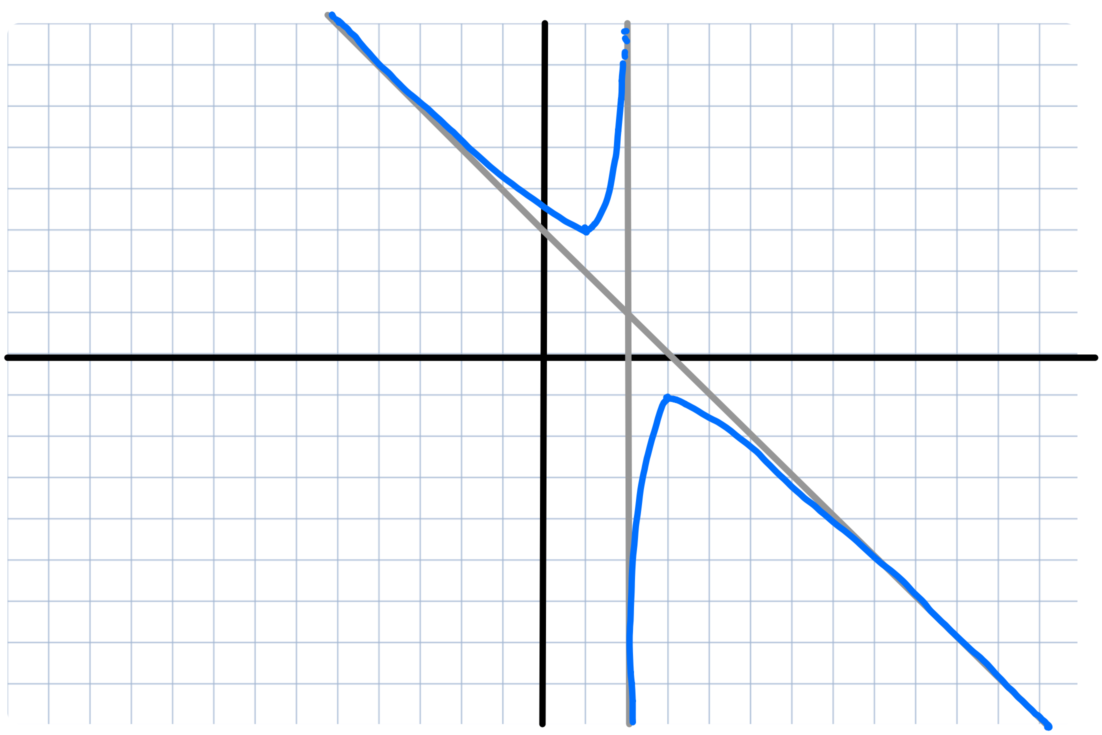
$f(1) = 3$ $\qquad$ $f(3) = -1$

$$f'' = \frac{(-2x+4)(x-2)^2 - (-x^2+4x-3)\cdot 2(x-2)}{(x-2)^4}$$

$$= \frac{(-2x+4)(x-2) - 2(-x^2+4x-3)}{(x-2)^3}$$

$$= \frac{-2}{(x-2)^3}$$

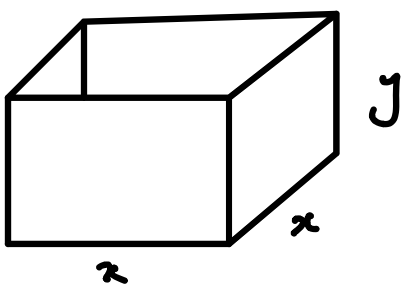
f'': $+$ on $(-\infty,2)$, $-$ on $(2,\infty)$



5. (25 points) A company needs to design an open topped box with a square base. The box must have volume 32in^3. If materials cost three dollars per square inch, what is the minimum possible cost of a single box?

Solution:

Objective : Minimize <u>Cost</u>



Objective quantity : $3(4xy + x^2)$

Constraint : $yx^2 = 32 \quad (x > 0)$

$$\Rightarrow \quad y = \frac{32}{x^2} \Rightarrow 3(4xy + x^2) = 3\left(\frac{128}{x} + x^2\right) = f(x)$$

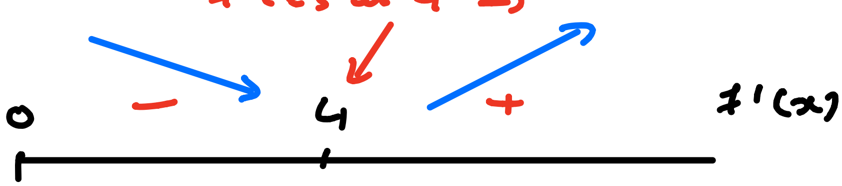
<u>Domain</u> : $(0, \infty)$

$$f'(x) = 3\left(\frac{-128}{x^2} + 2x\right)$$

A/ $\dfrac{-128}{x^2} + 2x = 0 \Leftrightarrow x^3 = 64 \Leftrightarrow x = 4$

B/ None in $(0, \infty)$

f cts at 4 $\Rightarrow$ Abs min.



$$\Rightarrow \text{Min cost} = f(4) = 3\left(\frac{128}{4} + 4^2\right) = \$144$$

END OF EXAM