MATH 1A MIDTERM 2 (001)
PROFESSOR PAULIN

DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.
CALCULATORS ARE NOT PERMITTED
YOU MAY USE YOUR OWN BLANK PAPER FOR ROUGH WORK
SO AS NOT TO DISTURB OTHER STUDENTS, EVERYONE MUST STAY UNTIL THE EXAM IS COMPLETE
REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT
THIS EXAM WILL BE ELECTRONICALLY SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED. YOU MAY WRITE SOLUTIONS ON THE BLANK PAGE AT THE BACK BUT BE SURE TO CLEARLY LABEL THEM

Name: ______________________________

Student ID: ______________________________

GSI's name: ______________________________

This exam consists of 5 questions. Answer the questions in the spaces provided.

1. Calculate the following:

   (a) (10 points)

$$\frac{d}{dx}(\frac{x^2-1}{2^x})$$

Solution:

$$\frac{d}{dx}\left(\frac{x^2-1}{2^x}\right) = \frac{\frac{d}{dx}(x^2-1)\cdot 2^x - (x^2-1)\frac{d}{dx}(2^x)}{(2^x)^2}$$

$$= \frac{2x\cdot 2^x - (x^2-1)\ln(2)\cdot 2^x}{(2^x)^2}$$

   (b) (15 points)

$$\lim_{x\to 0^+}(2x)^x$$

Solution:

$$\ln\left(\lim_{x\to 0^+}(2x)^x\right) = \lim_{x\to 0^+} x\ln(2x) = \lim_{x\to 0^+}\frac{\ln(2x)}{1/x}$$

L'Hospital

$$= \lim_{x\to 0^+}\frac{\frac{2}{2x}}{\frac{-1}{x^2}} = \lim_{x\to 0^+} -x = 0$$

$$\Rightarrow \lim_{x\to 0^+}(2x)^x = e^0 = 1$$

2. (25 points) Between November 13th 2019 and November 13th 2022, a radioactive material will lose 40% of its mass. Over what length of time will the material lose 80% of its mass? You do not need to simplify your answer.

Solution:

$t = 0$ 11/13/19
in years

$M(t) = Ce^{kt}$ = mass at time t

$M(3) = 0.6\,M(0)$

$\Rightarrow \quad Ce^{3k} = 0.6\,C \quad \Rightarrow \quad e^{3k} = 0.6$

$\Rightarrow \quad k = \dfrac{\ln(0.6)}{3}$

$M(t) = 0.2\,M(0) \quad \Rightarrow \quad Ce^{\frac{\ln(0.6)}{3}t} = 0.2C$

$\Rightarrow \quad \dfrac{\ln(0.6)}{3}t = \ln(0.2)$

$\Rightarrow \quad t = \dfrac{3\ln(0.2)}{\ln(0.6)}$

3. (25 points) Let f be a function which is differentiable on $\mathbb{R}$. Assume that

$$|f'(x)| < 2 \text{ for all } x \text{ in } \mathbb{R}$$

If $f(1) = 2$, what are the possible values of $f(3)$?

Solution:

M.V.T. $\Rightarrow$ $\dfrac{f(3)-f(1)}{3-1} = f'(c)$ for some c in $(1,3)$

$|f'(c)| < 2 \Rightarrow \left|\dfrac{f(3)-f(1)}{3-1}\right| < 2$

$\Rightarrow -2 < \dfrac{f(3)-2}{2} < 2$

$\Rightarrow -2 < f(3) < 6$

Conclusion : $f(3)$ can take any value in $(-2,6)$

4. (25 points) Sketch the following curve. Be sure to indicate asymptotes, local maxima and minima and concavity. Show your working on this page and draw the graph on the next page.

$$y = \frac{x^2+3x+3}{x+1} = f(x)$$

Solution:

<u>Domain</u> : $(-\infty,-1) \cup (-1,\infty)$

<u>Odd/Even</u> : Neither

<u>Vertical Asymptotes</u> :

$$\lim_{x \to -1^{+/-}} x^2+3x+3 = 1 > 0$$

$$\lim_{x \to -1^+} x+1 = 0^+ , \quad \lim_{x \to -1^-} x+1 = 0^- \quad \Bigg\} \quad \lim_{x \to -1^+} f(x) = \infty, \quad \lim_{x \to -1^-} f(x) = -\infty$$

<u>Behavior at $\pm\infty$</u> :

$$\lim_{x \to \pm\infty} \frac{f(x)}{x} = \lim_{x \to \pm\infty} \frac{x^2+3x+3}{x^2+x} = \frac{1}{1} = 1 = m$$

$$\lim_{x \to \pm\infty} f(x) - x = \lim_{x \to \pm\infty} \frac{\cancel{x^2}+3x+3-\cancel{x^2}-x}{x+1}$$

$$= \lim_{x \to \pm\infty} \frac{2x+3}{x+1} = 2 = b$$

$$f(x) = \frac{x^2+3x+3}{x+1}$$

A/ None $(x^2+3x+3 \neq 0)$

B/ $x = -1$

V.A.

f: $-$ | -1 | $+$

$f(-2) < 0$ $\quad$ $f(0) > 0$

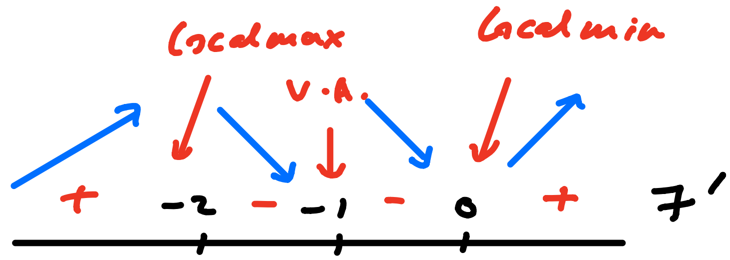
Solution (continued) :

$$f'(x) = \frac{(2x+3)(x+1) - (x^2+3x+3)}{(x+1)^2}$$

$$= \frac{x^2+2x}{(x+1)^2}$$

A/ $x^2+2x = 0 \iff x = 0, -2$

B/ $x = -1$

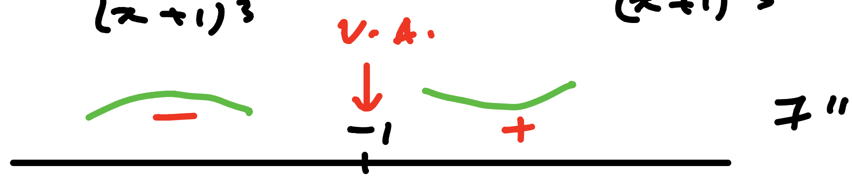


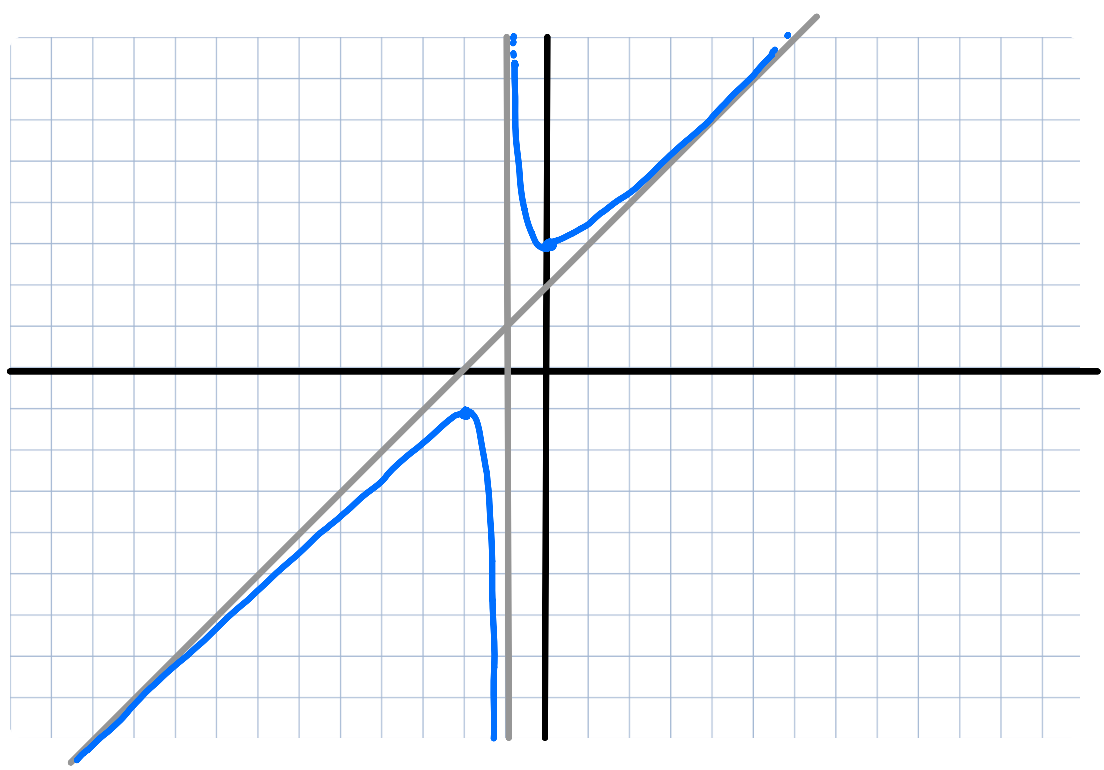
$$f''(x) = \frac{(2x+2)(x+1)^2 - (x^2+2x)\cdot 2(x+1)}{(x+1)^4}$$

$$= \frac{(2x+2)(x+1) - 2(x^2+2x)}{(x+1)^3} = \frac{2}{(x+1)^3}$$

A/ None

B/ $x = -1$





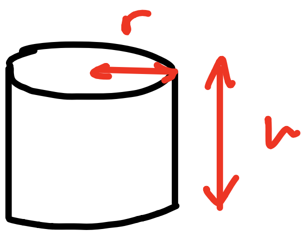
$f(0) = 3$

$f(-2) = -1$

5. (25 points) A company plans to package its product in a cylinder that is open at one end. The cylinder must have volume 27π cm^3. If materials cost 2 dollars per square centimeter, what is the minimum possible cost of producing a single cylinder? You do not need to simplify your answer.

Solution:

Objective : Minimize cost



Objective : $2(\pi r^2 + 2\pi r h)$

Constraint : Volume $= 27\pi$

$\Rightarrow \pi r^2 h = 27\pi$

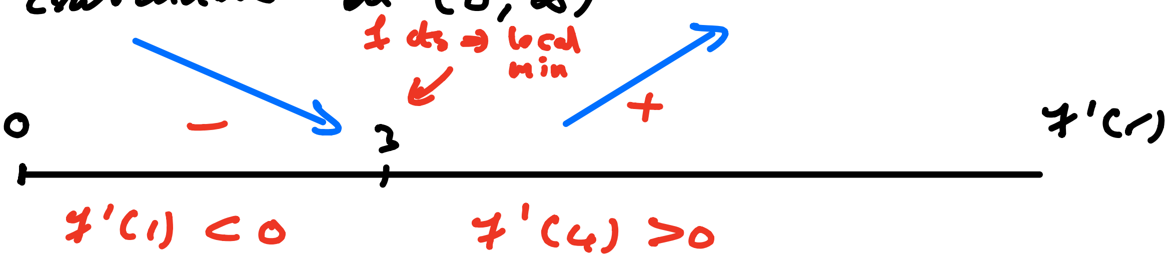
$\Rightarrow h = \dfrac{27}{r^2} \Rightarrow 2(\pi r^2 + 2\pi r h) = 2\left(\pi r^2 + \dfrac{54\pi}{r}\right) = f(r)$

Domain : $r \neq 0,\ r \geq 0 = (0, \infty)$

$f'(r) = 2\left(2\pi r - \dfrac{54\pi}{r^2}\right)$

A/ $f'(r) = 0 \Rightarrow r^3 = 27 \Rightarrow r = 3$

B/ f' continuous on $(0, \infty)$



$\Rightarrow f(3)$ absolute min on $(0, \infty)$

$\Rightarrow$ Min cost $= \$\, 2\left(\pi \cdot 3^2 + \dfrac{54\pi}{3}\right) = \$\, 54\pi$

END OF EXAM