Midterm 1 Review

Core Functions :

Polynomials, Rational Functions, Power Functions, Trigonometric Functions, Inverse Trigonometric Functions, Exponential Functions, Logarithmic Functions.

Review Definitions and basic properties

Domains

Domain of f = x in $\mathbb{R}$ such that $f(x)$ is defined.

Things to look out for :

- Denominators being 0
- Appropriate domains of core functions. For example

  arcsin, arccos have domain $[-1,1]$,

  ln has domain $(0,\infty)$

  integer

  tan has domain $x \neq \frac{\pi}{2} + n\pi$

  $\sqrt[n]{\ }$ for n even has domain $[0,\infty)$

- For compositions consider range of internal functions.

  For example, $\sqrt{\arcsin(x)}$

  $\arcsin(x) \geq 0 \iff 0 \leq x \leq 1$

  $\Rightarrow$ Domain of $\sqrt{\arcsin(x)} = [0,1]$

Transformations of Functions

$a > 0$

$f(x) \longrightarrow f(x) + a$ = Translation up by a

$f(x) \longrightarrow f(x) - a$ = Translation down by a

$f(x) \longrightarrow f(x+a)$ = Translation left by a

$f(x) \longrightarrow f(x-a)$ = Translation right by a

$b > 1$

$f(x) \longrightarrow b f(x)$ = Vertical stretch by b

$f(x) \longrightarrow \frac{1}{b} f(x)$ = Vertical compression by b

$f(x) \longrightarrow f(bx)$ = Horizontal compression by b

$f(x) \longrightarrow f(\frac{1}{b} x)$ = Horizontal stretch by b

$f(x) \longrightarrow -f(x)$ = Reflection in x-axis

$f(x) \longrightarrow f(-x)$ = Reflection in y-axis

Safe Order: ← Usually easiest to get correct.

Horizontal Translation → Horizontal Stretch / Compression → Reflection in y-axis

↓

Vertical Translation ← Reflection in x-axis ← Vertical stretch / Compression

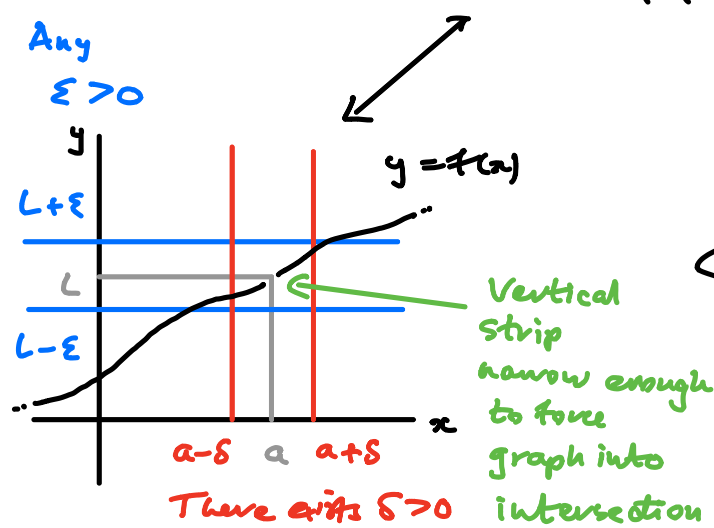
Limits

$$\lim_{x \to a} f(x) = L$$

Intuitive: $f(x)$ approaches L as x approaches (but does not equal) a from <u>both sides</u>.

Precise: Given <u>any</u> horizontal strip ① (centered at L), <u>there exists</u> ② a vertical strip (centered at a) such that

$(x, f(x))$ in ③ vertical strip, $x \neq a$ $\Rightarrow$ $(x, f(x))$ ④ in horizontal strip



$\Longleftrightarrow$ Given $\varepsilon > 0$, ① there exists $\delta > 0$, ② such that

$0 < |x - a| < \delta$ ③

$\Rightarrow |f(x) - L| < \varepsilon$ ④

Using (ε, δ)-definition to prove $\lim_{x \to a} f(x) = L$

1/ Fix $\varepsilon > 0$

2/ Draw Graph as above and make sensible choice of $\delta > 0$, i.e. Make vertical strip sufficiently narrow

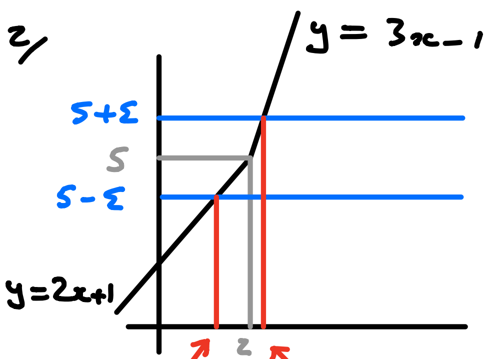
3/ Verify $0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$

Example $\quad f(x) = \begin{cases} 2x+1 & x \leq 2 \\ 3x-1 & x > 2 \end{cases}$

Prove $\lim_{x \to 2} f(x) = 5$

1/ Fix $\varepsilon > 0$

2/



$\Rightarrow$ Choose $\delta = \frac{\varepsilon}{3}$

$2x+1 = 5-\varepsilon$

$\Rightarrow x = 2 - \frac{\varepsilon}{2}$

$3x-1 = 5+\varepsilon$

$\Rightarrow x = 2 + \frac{\varepsilon}{3}$

smaller

3/ $0 < |x-2| < \frac{\varepsilon}{3}$

$x > 2$: $3|x-2| < \varepsilon \Rightarrow |3x-6| < \varepsilon$

$\Rightarrow |(3x-1)-5| < \varepsilon$

$x < 2$: $2|x-2| < \frac{2\varepsilon}{3} \Rightarrow |2x-4| < \frac{2\varepsilon}{3}$

$\Rightarrow |(2x+1)-5| < \frac{2\varepsilon}{3} < \varepsilon$

Remark: Trying to algebraically reverse engineer an appropriate $\delta > 0$ starting from $|f(x) - L| < \varepsilon$ will be very hard in this case.

Using (ε, δ) - definition to prove $\lim_{x \to a} f(x) \neq L$

1/ Draw a picture and choose $\varepsilon > 0$ small enough to divide $y = f(x)$ into two disjoint pieces near $x = a$

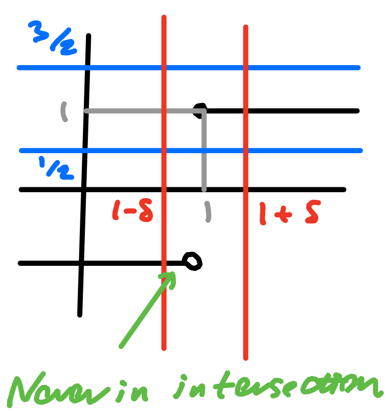
2/ For this ε, Prove that for <u>any</u> $\delta > 0$

$$0 < |x - a| < \delta \not\Rightarrow |f(x) - L| < \varepsilon$$

<u>Example</u> $f(x) = \begin{cases} -1 & \text{if } x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$

Prove $\lim_{x \to 1} f(x) \neq 1$

1/



$\Rightarrow$ Choose $\varepsilon = \frac{1}{2}$

2/ Let $\delta > 0$ and choose x such that $1 - \delta < x < 1$

$\Rightarrow$ $0 < |x - 1| < \delta$ <u>and</u> $|f(x) - 1| = |-1-1| = 2 \nless \frac{1}{2}$

$\Rightarrow$ No possible choice of $\delta > 0$ for $\varepsilon = \frac{1}{2}$

Important variants of $\lim_{x \to a} f(x) = L$:

$\lim_{x \to a^+} f(x) = L$, $\lim_{x \to a^-} f(x) = L$, $\lim_{x \to \infty} f(x) = L$, $\lim_{x \to -\infty} f(x) = L$

x approaches a from above

x approaches a from below

x grows positively without bound

Horizontal Asymptotes

x grows negatively without bound

Important Definition :

f continuous at $x = a$ if

1/ $f(a)$ defined

2/ $\lim_{x \to a} f(x)$ exists

3/ $\lim_{x \to a} f(x) = f(a)$

Variants : $\lim_{x \to a^+} f(x) = f(a)$, $\lim_{x \to a^-} f(x) = f(a)$

Continuous from above

Continuous from below

f is continuous if it is continous everywhere in its domain (above / below at end points)

Key Facts

1/ All core functions are continuous

2/ Sums , differences , products , quotients , compositions are continuous .

Limit Laws $\quad (? = a / a^+ / a^- / \infty, -\infty)$

$\lim_{x \to ?} f(x) = L$ and $\lim_{x \to ?} g(x) = K \quad \Rightarrow$

- $\lim_{x \to ?} f(x) + g(x) = L + K$
- $\lim_{x \to ?} f(x) g(x) = LK$
- $K \neq 0 \Rightarrow \lim_{x \to ?} \frac{f(x)}{g(x)} = \frac{L}{K}$

Composite Law :

$\lim_{x \to ?} g(x) = b \,,\ \lim_{x \to b} f(x) = c \Rightarrow \lim_{x \to ?} f(g(x)) = c$

Infinite Limits

$\lim_{x \to ?} f(x) = \infty \iff f(x)$ instead grows positively without bound

($a / a^+ / a^-$ gives vertical asymptotes)

$\lim_{x \to ?} f(x) = -\infty \iff f(x)$ instead grows negatively without bound

Warning : In both case Limit DNE.

Basic Laws of Infinite Limits $\quad (? = a / a^+ / a^- / \infty / -\infty)$

A/ $\lim_{x \to ?} f(x) = \infty \Rightarrow \lim_{x \to ?} -f(x) = -\infty$

B/ $\lim\limits_{x \to ?} f(x) = 0^{\pm} \iff \lim\limits_{x \to ?} \frac{1}{f(x)} = \pm\infty$

C/ $\lim\limits_{x \to ?} f(x) = L$, $\lim\limits_{x \to ?} g(x) = \pm\infty \Rightarrow \lim\limits_{x \to ?} f(x) + g(x) = \pm\infty$

D/ $\lim\limits_{x \to ?} f(x) = \infty$, $\lim\limits_{x \to ?} g(x) = \infty \Rightarrow \lim\limits_{x \to ?} f(x) + g(x) = \infty$

E/ $\lim\limits_{x \to ?} f(x) = L > 0$, $\lim\limits_{x \to ?} g(x) = \pm\infty \Rightarrow \lim\limits_{x \to ?} f(x)g(x) = \pm\infty$

F/ $\lim\limits_{x \to ?} f(x) = \infty$, $\lim\limits_{x \to ?} g(x) = \pm\infty \Rightarrow \lim\limits_{x \to ?} f(x)\, g(x) = \pm\infty$

G/ $\lim\limits_{x \to ?} g(x) = \pm\infty \Rightarrow \lim\limits_{x \to ?} f(g(x)) = \lim\limits_{u \to \pm\infty} f(u)$

Important Case : Quotients

$$\lim_{x \to ?} \frac{f(x)}{g(x)} = ???$$

$\downarrow$

Calculate $\lim\limits_{x \to ?} f(x) = L$

and $\lim\limits_{x \to ?} g(x) = K$

① $K \neq 0$ $\rightarrow$ Quotient Law

② $L \neq 0$, $K = 0$ $\rightarrow$ $\lim\limits_{x \to ?} \frac{f(x)}{g(x)}$ DNE

③ $L = 0$, $K = 0$ $\rightarrow$ Manipulate and apply ① or ②

②

Neither $\pm\infty$

$L > 0$, $K = 0^+$

$$\lim_{x \to ?} \frac{f(x)}{g(x)} = \infty$$

$L < 0$, $K = 0^-$

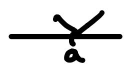
$$\lim_{x \to ?} \frac{f(x)}{g(x)} = \infty$$

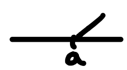
$L > 0$, $K = 0^-$

$$\lim_{x \to ?} \frac{f(x)}{g(x)} = -\infty$$

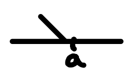
$L < 0$, $K = 0^+$

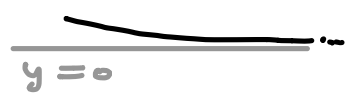
$$\lim_{x \to ?} \frac{f(x)}{g(x)} = -\infty$$

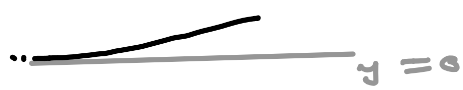
$\lim_{x \to a} g(x) = 0^+ \iff$ 

$\lim_{x \to a^+} g(x) = 0^+ \iff$ 

For 0^- reflect in y-axis

$\lim_{x \to a^-} g(x) = 0^+ \iff$ 

$\lim_{x \to \infty} g(x) = 0^+ \iff$ 

$\lim_{x \to \infty} g(x) = 0^+ \iff$ 

The Derivative

Double sided limit.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \text{slope of tangent line to graph at } (x, f(x))$$

If it exists we say f differentiable at x