MATH 1A MIDTERM 1 (PRACTICE 3)

PROFESSOR PAULIN

DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.
CALCULATORS ARE NOT PERMITTED
YOU MAY USE YOUR OWN BLANK PAPER FOR ROUGH WORK
SO AS NOT TO DISTURB OTHER STUDENTS, EVERYONE MUST STAY UNTIL THE EXAM IS COMPLETE
REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT
THIS EXAM WILL BE ELECTRONICALLY SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED. YOU MAY WRITE SOLUTIONS ON THE BLANK PAGE AT THE BACK BUT BE SURE TO CLEARLY LABEL THEM

Name and section: ______________________________

GSI's name: ______________________________

This exam consists of 5 questions. Answer the questions in the spaces provided.

1. Determine the domains of the following functions:

   (a) (15 points)

$$\sqrt{\frac{2-x}{(3-x)^3}}$$

Solution:

$$\begin{cases} 2-x \geq 0 \\ (3-x)^3 > 0 \end{cases} \Rightarrow \begin{cases} 2-x \geq 0 \\ 3-x \geq 0 \end{cases} \Rightarrow \begin{cases} 2 \geq x \\ 3 \geq x \end{cases} \Rightarrow 2 \geq x$$

$$\begin{cases} 2-x \leq 0 \\ (3-x)^3 < 0 \end{cases} \Rightarrow \begin{cases} 2-x \leq 0 \\ 3-x < 0 \end{cases} \Rightarrow \begin{cases} 2 \leq x \\ 3 < x \end{cases} \Rightarrow 3 < x$$

$\Rightarrow$ Domain is $(-\infty, 2] \cup (3, \infty)$

   (b) (10 points)

$$\arccos(7x+2)$$

Solution:

Domain of $\arccos(x) = [-1, 1]$

$-1 \leq 7x+2 \leq 1 \iff -3 \leq 7x \leq -1 \Rightarrow \frac{-3}{7} \leq x \leq \frac{-1}{7}$

$\Rightarrow$ Domain is $\left[\frac{-3}{7}, \frac{-1}{7}\right]$

2. (a) (15 points) Describe in words, how, starting with the graph $y = e^{x-2} - 2$, one can draw the graph

$$y = 2 + e^{-x}.$$

Solution:

$$e^{x-2} - 2 \xrightarrow{a)} e^{x} - 2 \xrightarrow{b)} e^{-x} - 2 \xrightarrow{c)} e^{-x} + 2$$

a) Translate to left by 2

b) Reflect in y-axis

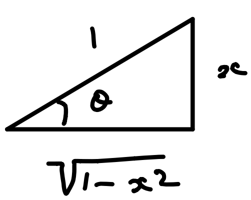
c) Translate up by 4

(b) (10 points) Express the following as an algebraic function:

$$\tan(\arcsin(x))$$

Solution:

Let $0 \le x < 1$ and $\theta = \arcsin(x) \Rightarrow 0 \le \theta < \frac{\pi}{2}$ and $\sin(\theta) = x$



$\Rightarrow \tan(\arcsin(x)) = \tan(\theta) = \dfrac{x}{\sqrt{1-x^2}}$

We've shown for x in $[0, 1)$

$\arcsin(x)$, $\tan(x)$ odd $\Rightarrow \tan(\arcsin(x))$ odd

$\dfrac{x}{\sqrt{1-x^2}}$ odd $\Rightarrow \tan(\arcsin(x)) = \dfrac{x}{\sqrt{1-x^2}}$ for all x in $(-1, 1)$

3. (25 points) Calculate (using the limit laws) the following limits. If a limit does not exist determine if it is ∞, $-\infty$ or neither.

(a)

$$\lim_{x\to 0}(\ln(x^4+1)+\cos(x))$$

Solution:

$$\lim_{x\to 0} x^4+1 = 0^4+1 = 1 \Rightarrow \lim_{x\to 0} \ln(x^4+1) = \ln(1) = 0$$

$$\lim_{x\to 0} \cos(x) = \cos(0) = 1$$

$$\Rightarrow \lim_{x\to 0}\left(\ln(x^4+1) + \cos(x)\right) = 0 + 1 = 1$$

(b)

$$\lim_{x\to 2^-}\frac{x^2+4x+2}{x^2-x-2}$$

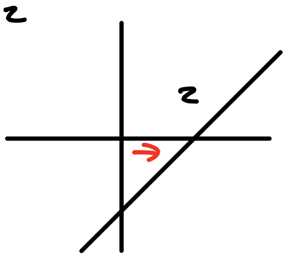
Solution:

$$\frac{x^2+4x+2}{x^2-x-2} = \frac{x^2+4x+2}{(x-2)(x+1)}$$

$$\lim_{x\to 2^-} x^2+4x+2 = 2^2+4\cdot 2+2 = 14 > 0$$

$$\lim_{x\to 2^-}(x+1) = 3 > 0 \qquad \Rightarrow \lim_{x\to 2^-}\frac{x^2+4x+2}{x^2-x-2} = -\infty$$

$$\lim_{x\to 2^-} x-2 = 0^-$$



(c)

$$\lim_{x\to 0^+} \arctan(\ln(x))$$

Solution:

$$\lim_{x\to 0^+} \ln(x) = -\infty$$

$$\lim_{u\to -\infty} \arctan(u) = -\frac{\pi}{2}$$

$$\Rightarrow \lim_{x\to 0^+} \arctan(\ln(x)) = \frac{-\pi}{2}$$

(d)

$$\lim_{x\to 0} x\sin(1/x^2)$$

Solution:

$$-1 \le \sin(1/x^2) \le 1$$

$$-x \le x\sin(1/x^2) \le x \quad (\text{if } x > 0)$$

$$-x \ge x\sin(1/x^2) \ge x \quad (\text{if } x < 0)$$

$$\lim_{x\to 0} -x = \lim_{x\to 0} x = 0 \Rightarrow \lim_{x\to 0^-} x\sin\left(\frac{1}{x^2}\right) = 0$$

<u>and</u>

$$\lim_{x\to 0^+} x\sin(1/x^2) = 0$$

$$\Rightarrow \lim_{x\to 0} x\sin(1/x^2) = 0$$

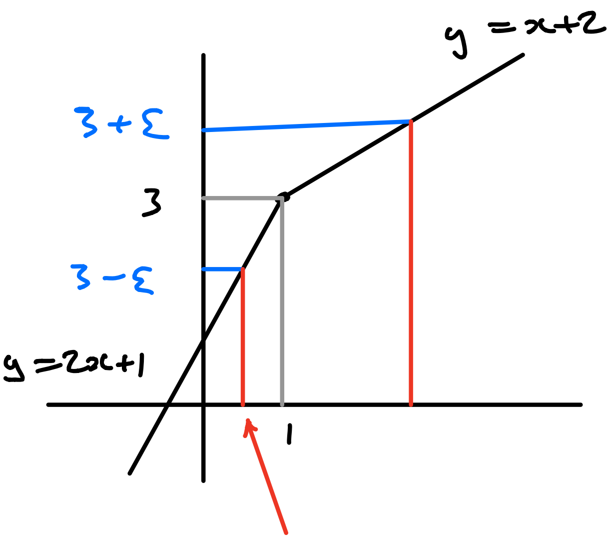
4. (25 points) Let $f(x) = \begin{cases} 2x+1 & \text{if } x \le 1 \\ x+2 & \text{if } x > 1 \end{cases}$.

Prove, using ϵ, δ methods, that

$$\lim_{x \to 1} f(x) = 3.$$

Solution:

Let $\varepsilon > 0$



$2x+1 = 3 - \varepsilon$

$\Rightarrow \quad x = 1 - \frac{\varepsilon}{2}$

Let $\delta = \frac{\varepsilon}{2}$

$0 < |x-1| < \frac{\varepsilon}{2} \xrightarrow{x<1} 2|x-1| < \varepsilon \Rightarrow |2(x-1)| < \varepsilon$

$\Rightarrow |(2x+1) - 3| < \varepsilon$

$0 < |x-1| < \frac{\varepsilon}{2} \xrightarrow{x>1} |(x+2) - 3| < \frac{\varepsilon}{2} < \varepsilon$

$\Rightarrow \quad \lim_{x \to 1} f(x) = 3$

5. (25 points) Let $f(x) = \begin{cases} x^2 & \text{if } x \le 1 \\ \frac{1}{\sqrt{x}} & \text{if } x > 1 \end{cases}$

Using the direct definition of the derivative, show that f is **not** differentiable at $x = 1$.

Solution:

$$\lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^+} \frac{\frac{1}{\sqrt{1+h}} - 1}{h}$$

$$= \lim_{h \to 0^+} \frac{1 - \sqrt{1+h}}{h\sqrt{1+h}} = \lim_{h \to 0^+} \frac{(1-\sqrt{1+h})(1+\sqrt{1+h})}{h\sqrt{1+h}\,(1+\sqrt{1+h})}$$

$$= \lim_{h \to 0^+} \frac{-1}{\sqrt{1+h}\,(1+\sqrt{1+h})} = \frac{-1}{2}$$

$$\lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h} = \lim_{h \to 0^-} \frac{(1+h)^2 - 1}{h} = \lim_{h \to 0^-} \frac{h^2 + 2h}{h}$$

$$= \lim_{h \to 0^-} h + 2 = 2$$

$$\Rightarrow \lim_{h \to 0^+} \frac{f(1+h) - f(1)}{h} \neq \lim_{h \to 0^-} \frac{f(1+h) - f(1)}{h}$$

$$\Rightarrow \lim_{h \to 0} \frac{f(1+h) - f(1)}{h} \text{ DNE} \Rightarrow f \text{ \underline{not} differentiable at } x = 1$$

END OF EXAM