MATH 1A MIDTERM 1 (PRACTICE 1)

PROFESSOR PAULIN

DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.

CALCULATORS ARE NOT PERMITTED

YOU MAY USE YOUR OWN BLANK PAPER FOR ROUGH WORK

SO AS NOT TO DISTURB OTHER STUDENTS, EVERYONE MUST STAY UNTIL THE EXAM IS COMPLETE

REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT

THIS EXAM WILL BE ELECTRONICALLY SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED. YOU MAY WRITE SOLUTIONS ON THE BLANK PAGE AT THE BACK BUT BE SURE TO CLEARLY LABEL THEM

Name and section: ______________________________

GSI's name: ______________________________

This exam consists of 5 questions. Answer the questions in the spaces provided.

1. Determine the domains of the following functions:

(a) (10 points)

$$\frac{\sqrt{1-x^2}}{\tan^2(x)-1}$$

Solution:

$1-x^2 \geq 0 \iff 1 \geq x^2 \iff 1 \leq x \leq 1$, i.e. x in $[-1,1]$

$\tan^2(x)-1 = 0 \iff \tan(x) = \pm 1$. In $[-1,1]$ this only occurs at $\pm \frac{\pi}{4}$

$\Rightarrow$ Domain $= [-1, \frac{-\pi}{4}) \cup (\frac{-\pi}{4}, \frac{\pi}{4}) \cup (\frac{\pi}{4}, 1]$

(b) (15 points)

$$\ln(\sin(2x)+1)$$

Solution:

$\sin(2x)+1 \geq 0$ for all x in $\mathbb{R}$

$\sin(2x)+1 = 0 \iff \sin(2x) = -1$

$\iff 2x = \frac{-\pi}{2} + 2k\pi$ (k an integer)

$\iff x = \frac{-\pi}{4} + k\pi \Rightarrow$ Domain is all x in $\mathbb{R}$ such that $x \neq \frac{-\pi}{4} + k\pi$, for k an integer

2. (a) (15 points) Describe in words, how, starting with the graph $y = f(x)$, one can draw the graph

$$y = -3f(\frac{2-x}{3}) + 1.$$

Solution:

$$f(x) \xrightarrow{a)} f\left(\frac{x}{3}\right) \xrightarrow{b)} f\left(\frac{x+2}{3}\right) \xrightarrow{c)} f\left(\frac{-x+2}{3}\right)$$

$$\downarrow d)$$

$$-3f\left(\frac{2-x}{3}\right)+1 \xleftarrow{f)} -3f\left(\frac{-x+2}{3}\right) \xleftarrow{e)} 3f\left(\frac{-x+2}{3}\right)$$

a) Stretch horizontally by 3

b) translate to left by 2

c) Reflect in y-axis

d) stretch vertically by 3

e) Reflect in x-axis

f) translate up by 1

(b) (10 points) Give the precise value of the following:

$$\arctan(\tan(\frac{19\pi}{6}))$$

Solution:

in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\tan\left(\frac{19\pi}{6}\right) = \tan\left(3\pi + \frac{\pi}{6}\right) = \tan\left(\frac{\pi}{6}\right)$$

$$\Rightarrow \arctan\left(\tan\left(\frac{19\pi}{6}\right)\right) = \arctan\left(\tan\left(\frac{\pi}{6}\right)\right) = \frac{\pi}{6}$$

3. (25 points) Calculate (using the limit laws) the following limits. If a limit does not exist determine if it is ∞, $-\infty$ or neither.

(a)

$$\lim_{x \to 0} \sin(\pi(x^2+1))$$

Solution:

Polynomials are continuous

$$\lim_{x \to 0} \pi(x^2+1) = \pi(0^2+1) = \pi$$

$\sin(x)$ continuous at $x = \pi$

$$\Rightarrow \lim_{x \to 0} \sin(\pi(x^2+1)) = \sin(\pi) = 0$$

(b)

$$\lim_{x \to -2} \frac{x^2-x-6}{x^2+4x+4}$$

Solution:

$x \neq -2$

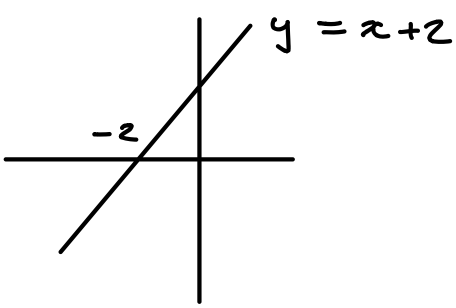
$$\frac{x^2-x-6}{x^2+4x+4} = \frac{(x+2)(x-3)}{(x+2)(x+2)} = \frac{(x-3)}{x+2}$$

$$\lim_{x \to -2} x-3 = -2-3 = -5 < 0$$

$$\lim_{x \to -2} x+2 = -2+2 = 0$$

$$\lim_{x \to -2} \frac{x^2-x-6}{x^2+4x+4} \text{ DNE}$$

$\Uparrow$ and $\neq \pm\infty$



$$\Rightarrow \lim_{x \to -2^+} x+2 = 0^+ \qquad \lim_{x \to -2^-} x+2 = 0^-$$

$$\Rightarrow \lim_{x \to -2^+} \frac{x-3}{x+2} = -\infty \qquad \lim_{x \to -2^-} \frac{x-3}{x+2} = \infty$$

(c)

$$\lim_{x\to 1} \arccos\left(\frac{1-\sqrt{x}}{1-x}\right)$$

Solution:

$x \neq 1$

$$\frac{1-\sqrt{x}}{1-x} = \frac{(1-\sqrt{x})(1+\sqrt{x})}{(1-x)(1+\sqrt{x})} = \frac{1-x}{(1-x)(1+\sqrt{x})} = \frac{1}{1+\sqrt{x}}$$

$$\lim_{x\to 1} 1 = 1, \quad \lim_{x\to 1} 1+\sqrt{x} = 1+\sqrt{1} = 2 \Rightarrow \lim_{x\to 1} \frac{1}{1+\sqrt{x}} = \frac{1}{2}$$

$\arccos(x)$ continuous at $x = \frac{1}{2}$

$$\Rightarrow \lim_{x\to 1} \arccos\left(\frac{1-\sqrt{x}}{1-x}\right) = \arccos\left(\frac{1}{2}\right) = \frac{\pi}{3}$$

(d)

$$\lim_{x\to -\infty} \frac{2x+7}{\sqrt{x^2+9}}$$

Solution:

Let $x<0$ and divide top and bottom by $-x = \sqrt{x^2}$

$$\Rightarrow \lim_{x\to -\infty} \frac{2x+7}{\sqrt{x^2+9}} = \lim_{x\to -\infty} \frac{-2-\frac{7}{x}}{\sqrt{1+\frac{9}{x^2}}}$$

$$\lim_{x\to -\infty} -2-\frac{7}{x} = -2$$

$$\lim_{x\to -\infty} \sqrt{1+\frac{9}{x^2}} = \sqrt{1} = 1$$

$$\Rightarrow \lim_{x\to -\infty} \frac{2x+7}{\sqrt{x^2+9}} = -2$$

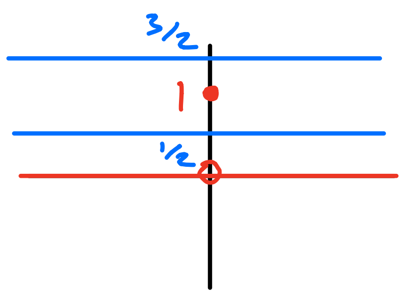
4. (25 points) Prove, using ϵ, δ methods, that the following function is **not** continuous at $x = 0$:

$$f(x) = \begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$$

Need to show $\lim_{x \to 0} f(x) \neq 1$

Let $\varepsilon = \frac{1}{2}$. We'll show we cannot find $\delta > 0$

such that $0 < |x| < \delta \Rightarrow |f(x) - 1| < \frac{1}{2}$



Note that $0 < |x| < \delta \Rightarrow x \neq 0 \Rightarrow f(x) = 0$

$\Rightarrow |f(x) - 1| = |0 - 1| = 1 > \frac{1}{2}$

Hence for $\varepsilon = \frac{1}{2}$ there is no possible $\delta > 0$.

$\Rightarrow \lim_{x \to 0} f(x) \neq 1 \Rightarrow f$ not continuous at $x = 0$

5. (a) (15 points) Using the direct definition of the derivative to calculate the derivative of the function

$$f(x) = \sqrt{2-x}.$$

What is the domain of the $f'(x)$?

Solution:

$$f'(x) = \lim_{h\to 0} \frac{\sqrt{2-(x+h)} - \sqrt{2-x}}{h}$$

$$= \lim_{h\to 0} \frac{\left(\sqrt{2-(x+h)} - \sqrt{2-x}\right)\left(\sqrt{2-(x+h)} + \sqrt{2-x}\right)}{h \cdot \left(\sqrt{2-(x+h)} + \sqrt{2+x}\right)}$$

$$= \lim_{h\to 0} \frac{-h}{h\left(\sqrt{2-(x+h)} + \sqrt{2-x}\right)} = \frac{-1}{2\sqrt{2-x}}$$

Domain is all x such that $2-x>0 \iff 2>x$

i.e. all x in $(-\infty, 2)$

(b) (10 points) Show that the line $y = \frac{-x+3}{2}$ is a tangent line to some point on the graph $y = f(x)$.

Solution:

$$f'(x) = \frac{-1}{2} \Rightarrow \frac{-1}{2\sqrt{2-x}} = \frac{-1}{2} \Rightarrow \sqrt{2-x} = 1 \Rightarrow 2-x=1$$

$$\Rightarrow x = 1$$

$f(1) = \sqrt{2-1} = 1 \Rightarrow$ Tangent line at $x=1$ is

$$y-1 = \frac{-1}{2}(x-1) \Rightarrow y = \frac{-x+3}{2}$$

END OF EXAM