MATH 1A MIDTERM 1 (002)
PROFESSOR PAULIN

DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.

CALCULATORS ARE NOT PERMITTED

YOU MAY USE YOUR OWN BLANK PAPER FOR ROUGH WORK

SO AS NOT TO DISTURB OTHER STUDENTS, EVERYONE MUST STAY UNTIL THE EXAM IS COMPLETE

REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT

THIS EXAM WILL BE ELECTRONICALLY SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED. YOU MAY WRITE SOLUTIONS ON THE BLANK PAGE AT THE BACK BUT BE SURE TO CLEARLY LABEL THEM

Name: ______________________________

Student ID: ______________________________

GSI's name: ______________________________

This exam consists of 5 questions. Answer the questions in the spaces provided.

1. Determine the domains of the following functions:

   (a) (15 points)

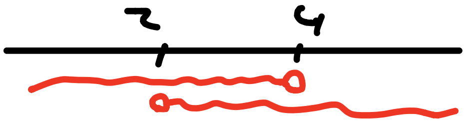
$$\frac{x+1}{\sqrt{\frac{x-4}{2-x}}}$$

Solution:

$\frac{x-4}{2-x} > 0$

$\frac{+}{+} \rightarrow$ $x-4>0$, $2-x>0$ $\iff$ $x>4$, $2>x$ $\Rightarrow$ No solutions

$\frac{-}{-} \rightarrow$ $x-4<0$, $2-x<0$ $\iff$ $x<4$, $2<x$ $\iff$ x in $(2,4)$



$\Rightarrow$ Domain of $\dfrac{x+1}{\sqrt{\frac{x-4}{2-x}}}$

   (b) (10 points)

$$\ln(-\arcsin(x))$$

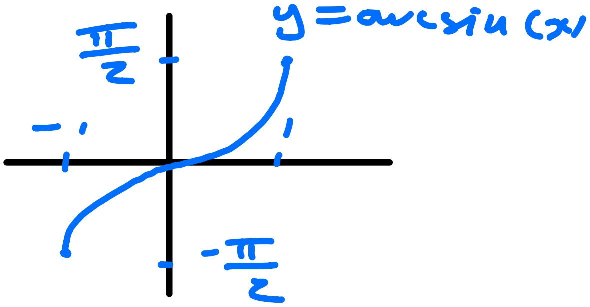
Solution:

$-\arcsin(x) > 0 \iff \arcsin(x) < 0$

$\iff x$ in $[-1, 0)$

closed

open



Hence domain of $\ln(-\arcsin(x))$ is $[-1, 0)$

2. (a) (15 points) Describe in words, how, starting with the graph $y = -\tan(2x+1) - 1$, one can draw the graph

$$y = \tan(2 - x) - 2.$$

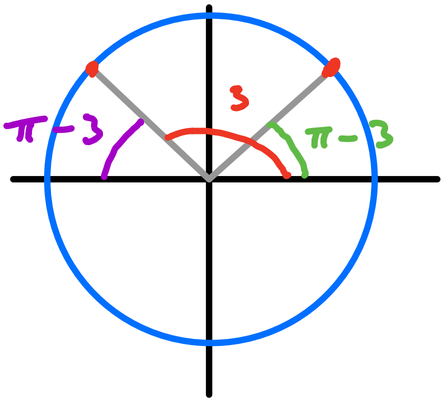
Solution:

$$\begin{tikzcd}
y = -\tan(2x+1) - 1 \arrow[r, "\text{stretch horizontally by factor 2}"] & y = -\tan(x+1) - 1 \arrow[d, "\text{Translate left by 1}"] \\
y = -\tan(2-x) - 1 \arrow[d, "\text{Reflect in } x\text{-axis}"'] & y = -\tan(x+2) - 1 \arrow[l, "\text{Reflect in } y\text{-axis}"'] \\
y = \tan(2-x) + 1 \arrow[r, "\text{Translate down by 3}"] & y = \tan(2-x) - 2
\end{tikzcd}$$

(b) (10 points) Simplify the following expression:

$$\arcsin(\sin(3))$$

Solution:



$$\sin(3) = \sin(\overset{\overset{\text{In } [-\frac{\pi}{2}, \frac{\pi}{2}]}{\downarrow}}{\pi - 3})$$

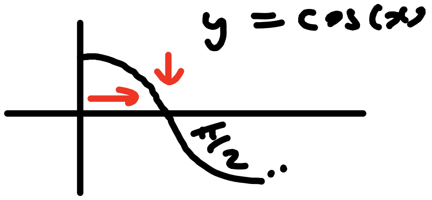
$$\Rightarrow \arcsin(\sin(3)) = \pi - 3$$

3. Calculate (using the limit laws) the following limits. If a limit does not exist determine if it is ∞, $-\infty$ or neither.

   (a) (15 points)

$$\lim_{x\to\pi/2^-} x\sec(x)$$

Solution:



$$x\sec(x) = \frac{x}{\cos(x)}$$

$$\lim_{x\to\frac{\pi}{2}^-} x = \frac{\pi}{2} > 0$$

$$\lim_{x\to\frac{\pi}{2}^-} \cos(x) = 0^+$$

$$\Rightarrow \lim_{x\to\frac{\pi}{2}^-} \frac{x}{\cos(x)} = \infty$$

   (b) (10 points)

$$\lim_{x\to\infty} \ln\left(\frac{x^3+1}{x^4+x+1}\right)$$

Solution:

$$\deg(x^3+1) = 3 < 4 = \deg(x^4+x+1)$$

$$\lim_{x\to\infty} \frac{x^3+1}{x^4+x+1} = 0^+$$

$$\lim_{x\to 0^+} \ln(x) = -\infty$$

$$\Rightarrow \lim_{x\to\infty} \ln\left(\frac{x^3+1}{x^4+x+1}\right) = -\infty$$

4. (25 points) Let $f(x) = \begin{cases} 1-2x & \text{if } x \leq 2 \\ -3 & \text{if } x > 2 \end{cases}$.

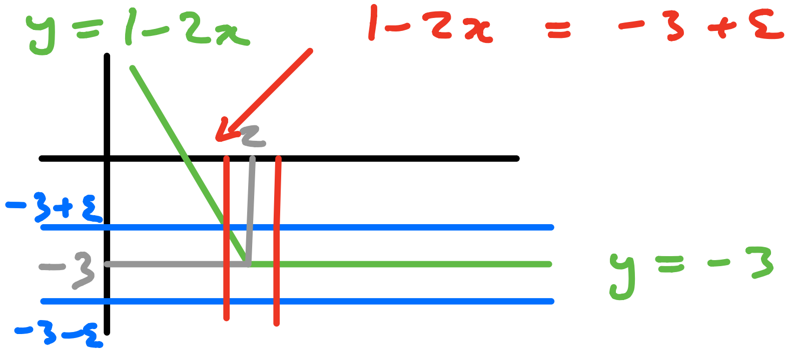
Prove, using ϵ, δ methods, that f is continuous at $x = 2$.

Solution:

Claim $\lim_{x \to 2} f(x) = f(2) = -3$

Fix $\varepsilon > 0$

$1-2x = -3+\varepsilon \Rightarrow x = 2 - \frac{\varepsilon}{2}$



$\Rightarrow$ Choose $\delta = \frac{\varepsilon}{2}$

$|4-2x| = |2x-4|$

||

$0 < |x-2| < \frac{\varepsilon}{2} \Rightarrow 2|x-2| < \varepsilon \Rightarrow |2x-4| < \varepsilon \Rightarrow |1-2x-(-3)| < \varepsilon$

↑ $x \leq 2$ ↑ $x \leq 2$

$0 < |x-2| < \frac{\varepsilon}{2}$ $|f(x)-(-3)| < \varepsilon$

↓ $x > 2$

$|-3-(-3)| = 0 < \varepsilon$

always true so nothing to check

Hence $\lim_{x \to 2} f(x) = f(2) = -3$

5. (25 points) Does there exist a tangent line to the curve

$$y = \frac{2-x}{x-1}$$

which contains the point $(1, 0)$? Carefully justify your answer. If you calculate a derivative do so using the limit definition. You do not need to use ϵ, δ methods.

Solution:

$$f(x) = \frac{2-x}{x-1}$$

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \lim_{h \to 0} \frac{\frac{2-a-h}{a+h-1} - \frac{2-a}{a-1}}{h}$$

$$= \lim_{h \to 0} \frac{(2-a-h)(a-1) - (2-a)(a+h-1)}{h(a+h-1)(a-1)}$$

$$= \lim_{h \to 0} \frac{\cancel{2a} - \cancel{2} - \cancel{a^2} + \cancel{a} - \cancel{ha} + h - \cancel{2a} - 2h + \cancel{2} + \cancel{a^2} + \cancel{ah} - \cancel{a}}{h(a+h-1)(a-1)}$$

$$= \lim_{h \to 0} \frac{-1}{(a+h-1)(a-1)} = \frac{-1}{(a-1)^2}$$

Equation of tangent at $x = a$: $y - \frac{2-a}{a-1} = \frac{-1}{(a-1)^2}(x-a)$

$(1,0)$ on tangent $\iff -\frac{2-a}{a-1} = \frac{-1}{(a-1)^2}(1-a)$

$\iff \frac{a-2}{a-1} = \frac{1}{a-1} \iff a-2 = 1$

$\iff a = 3$

$\Rightarrow$ There is precisely one tangent line containing $(1,0)$. It is at $x = 3$.

END OF EXAM