Derivatives of Logarithmic Functions

Let $b > 0$, $b \neq 1$

Recall: $\quad y = \log_b(x) \iff b^y = x$

$$\Rightarrow \frac{d}{dx}(b^y) = \frac{d}{dx}(x)$$

$$\Rightarrow \ln(b)\, b^y \cdot \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{\ln(b)\, b^y} = \frac{1}{\ln(b)\, x}$$

Conclusion:

$$\boxed{\frac{d}{dx}(\log_b(x)) = \frac{1}{\ln(b)\, x}}$$

$$\ln(e) = 1 \Rightarrow \boxed{\frac{d}{dx}(\ln(x)) = \frac{1}{x}}$$

Often called relative derivative

$$\text{Chain Rule} \Rightarrow \boxed{\frac{d}{dx}(\ln(g(x))) = \frac{g'(x)}{g(x)}}$$

Example

1/ $\frac{d}{dx}(\ln(|x|)) = ?$

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$$\frac{d}{dx}(\ln(x)) = \frac{1}{x}, \quad \frac{d}{dx}(\ln(-x)) = \frac{-1}{-x} = \frac{1}{x}$$

$$\Rightarrow \quad \frac{d}{dx}\left(\ln(|x|)\right) = \frac{1}{x}$$

2/ $\frac{d}{dx}\left(\log_3(\cos(x)+1)\right) = ?$

Chain Rule

$$\frac{d}{dx}\left(\log_3(\cos(x)+1)\right) = \frac{1}{\ln(3)(\cos(x)+1)} \cdot (-\sin(x))$$

Logarithmic Differentiation

$$\frac{d}{dx}(x^x) = ?$$

Problem : x^x is not of form b^x or x^r.

Clever Solution :

$$y = x^x \Rightarrow \ln(y) = \ln(x^x) = x\ln(x)$$

Implicit Differentiation

$$\Rightarrow \quad \frac{d}{dx}(\ln(y)) = \frac{d}{dx}(x\ln(x))$$

$$\Rightarrow \quad \frac{1}{y}\frac{dy}{dx} = \frac{d}{dx}(x)\ln(x) + x\frac{d}{dx}(\ln(x)) = \ln(x) + 1$$

$$\Rightarrow \quad \frac{dy}{dx} = y(\ln(x)+1) = x^x(\ln(x)+1)$$

Overview

$$\frac{d}{dx}(\ln(f(x))) = \frac{f'(x)}{f(x)}$$

$$\Rightarrow \quad f'(x) = f(x)\underbrace{\frac{d}{dx}\ln(f(x))}_{\text{May be easier to calculate}}$$

Examples

1/ $y = \dfrac{\sqrt{x}\,\sqrt[3]{x^2+1}}{(3x+5)^5} \Rightarrow \dfrac{dy}{dx} = ?$

Logarithm Laws $\Rightarrow$

$$\ln(y) = \frac{1}{2}\ln(x) + \frac{1}{3}\ln(x^2+1) - 5\ln(3x+5)$$

$$\Rightarrow \frac{d}{dx}\ln(y) = \frac{1}{2x} + \frac{2x}{3(x^2+1)} - \frac{5\times 3}{3x+5}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{1}{2x} + \frac{2x}{3(x^2+1)} - \frac{5\times 3}{3x+5}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\sqrt{x}\,\sqrt[3]{x^2+1}}{(3x+5)^5}\left(\frac{1}{2x} + \frac{2x}{3(x^2+1)} - \frac{5\times 3}{3x+5}\right)$$

2/ $y = x^r \Rightarrow \ln(y) = r\ln(x)$

$$\Rightarrow \frac{d}{dx}(\ln(y)) = \frac{d}{dx}(r\ln(x))$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{r}{x} \Rightarrow \frac{dy}{dx} = \frac{r}{x}\cdot y = \frac{r}{x}x^r = rx^{r-1}$$

$$\Rightarrow \frac{d}{dx}(x^r) = rx^{r-1}$$ Power Rule

Observations

$f(x) = \ln(x) \Rightarrow f'(x) = \dfrac{1}{x} \Rightarrow f'(1) = 1$

$$\Rightarrow \lim_{h\to 0} \frac{\ln(1+h) - \ln(1)}{h} = 1$$

$$\Rightarrow \lim_{h \to 0} \frac{\ln(1+h)}{h} = 1$$

Logarithm Laws

$$\Rightarrow \lim_{h \to 0} \ln\left((1+h)^{1/h}\right) = 1$$

$$\ln(e) = 1 \Rightarrow \boxed{\lim_{h \to 0} (1+h)^{1/h} = e}$$

$$x = \frac{1}{h}$$

$$h \to 0^+ \Rightarrow x \to \infty$$

Hence $$\boxed{\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x = e}$$