Limits at Infinity

Definition

"x grows positively without bound"

Limit

$\lim_{x \to \infty} f(x) = L$ or $f(x) \to L$ as $x \to \infty$

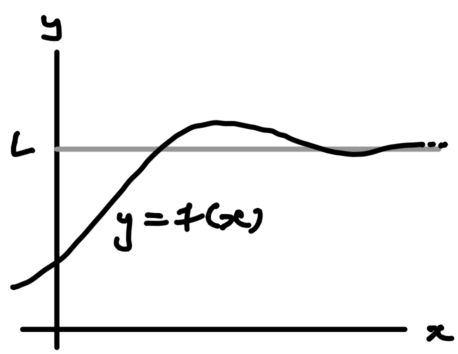
$\iff$ <u>$f(x)$ approaches L</u> as x <u>grows positively</u> <u>without bound.</u>

"x grows negatively without bound"

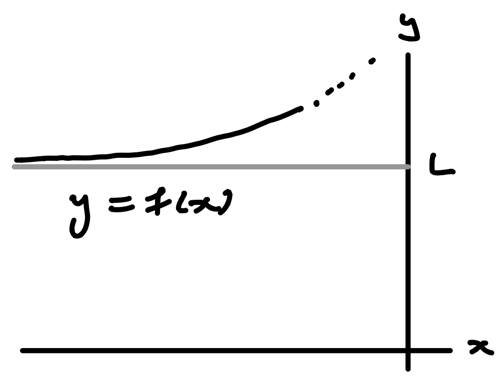
$\lim_{x \to -\infty} f(x) = L$ or $f(x) \to L$ as $x \to -\infty$

$\iff$ <u>$f(x)$ approaches L</u> as x <u>grows negatively</u> <u>without bound.</u>

Examples



$\lim_{x \to \infty} f(x) = L$



$\lim_{x \to -\infty} f(x) = L$

In either case we say $y = L$ is a <u>horizontal asymptote</u>

Important Examples

$$\lim_{x \to \infty} \arctan(x) = \frac{\pi}{2}, \quad \lim_{x \to -\infty} \arctan(x) = \frac{-\pi}{2}$$

$$\lim_{x \to -\infty} e^{x} = 0,$$

$$r > 0 \Rightarrow \lim_{x \to \infty} \frac{1}{x^r} = 0$$

x^r defined for $x < 0 \Rightarrow \lim_{x \to -\infty} \frac{1}{x^r} = 0$

Fact:

All usual limit laws (and squeeze) hold for limits at infinity. Here's the quotient law:

$$\lim_{x \to \infty} f(x) = L, \quad \lim_{x \to \infty} g(x) = K \neq 0 \quad \Rightarrow \quad \lim_{x \to \infty} \frac{f(x)}{g(x)} = \frac{L}{K}$$

Example

divide top and bottom by x^2

$$\lim_{x \to \infty} \frac{2x^2+1}{3x^2+x+1} = \lim_{x \to \infty} \frac{2 + \frac{1}{x^2}}{3 + \frac{1}{x} + \frac{1}{x^2}}$$

$$= \frac{\lim_{x \to \infty} 2 + \lim_{x \to \infty} \frac{1}{x^2}}{\lim_{x \to \infty} 3 + \lim_{x \to \infty} \frac{1}{x} + \lim_{x \to \infty} \frac{1}{x^2}} = \frac{2}{3}$$

Fact :

$$\lim_{x \to \pm\infty} \frac{p(x)}{q(x)} = \begin{cases} 0 & \text{if } \deg(p(x)) < \deg(q(x)) \\ \text{Ratio of leading coefficients} & \text{if } \deg(p(x)) = \deg(q(x)) \\ \text{DNE} & \text{if } \deg(p(x)) > \deg(q(x)) \end{cases}$$

Examples

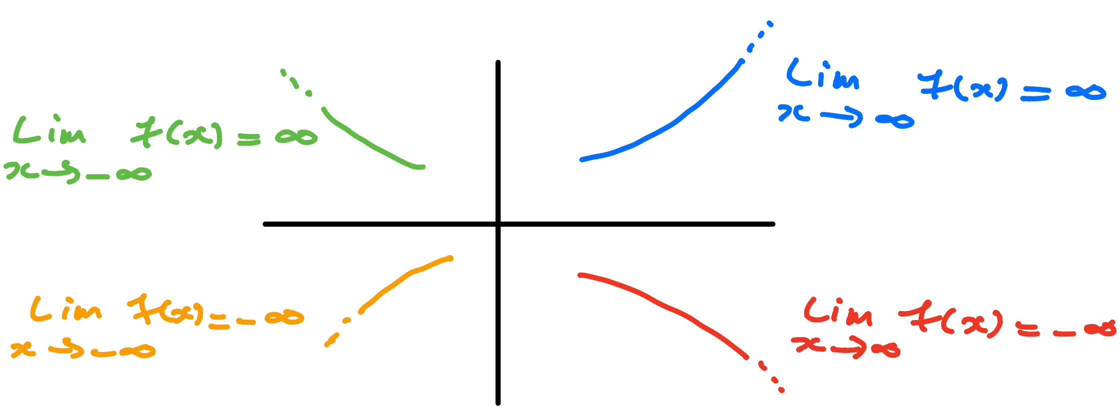
$$\lim_{x \to \pm\infty} \frac{x+1}{x^3+1} = 0 \,, \quad \lim_{x \to \pm\infty} \frac{3x^7 - 1}{4x^7 + x} = \frac{3}{4} \,, \quad \lim_{x \to \pm\infty} \frac{x^2+1}{x+1} \text{ DNE}$$

Infinite Limits at Infinity

$$\lim_{x \to \infty} f(x) = \infty \quad \text{or} \quad f(x) \to \infty \quad \text{as } x \to \infty$$

$\iff$ $f(x)$ grows positively without bound as x grows positively without bound.

We have similar definitions for $\lim_{x \to \infty} f(x) = -\infty$, $\lim_{x \to -\infty} f(x) = \infty$ and $\lim_{x \to -\infty} f(x) = -\infty$



Important Examples

$\lim_{x \to \infty} e^x = \infty$, $\lim_{x \to \infty} \ln(x) = \infty$, $\lim_{x \to \infty} x^r = \infty$ (if $r > 0$)

Useful Facts about Infinite Limits ($? = a / a^+ / a^- / \infty / -\infty$)

A/ $\lim_{x \to ?} f(x) = \infty \Rightarrow \lim_{x \to ?} -f(x) = -\infty$

B/ $\lim_{x \to ?} f(x) = \pm\infty \Leftrightarrow \lim_{x \to ?} \frac{1}{f(x)} = 0^{\pm}$

C/ $\lim_{x \to ?} f(x) = L$, $\lim_{x \to ?} g(x) = \pm\infty \Rightarrow \lim_{x \to ?} f(x) + g(x) = \pm\infty$

D/ $\lim_{x \to ?} f(x) = \infty$, $\lim_{x \to ?} g(x) = \infty \Rightarrow \lim_{x \to ?} f(x) + g(x) = \infty$

E/ $\lim_{x \to ?} f(x) = L > 0$, $\lim_{x \to ?} g(x) = \pm\infty \Rightarrow \lim_{x \to ?} f(x)g(x) = \pm\infty$

F/ $\lim_{x \to ?} f(x) = \infty$, $\lim_{x \to ?} g(x) = \pm\infty \Rightarrow \lim_{x \to ?} f(x)g(x) = \pm\infty$

G/ $\lim_{x \to ?} g(x) = \pm\infty \Rightarrow \lim_{x \to ?} f(g(x)) = \lim_{u \to \pm\infty} f(u)$

Examples

1/ $\lim_{x \to \infty} x^2 - x = ?$

WRONG: $\lim_{x \to \infty} x^2 - x = \lim_{x \to \infty} x^2 + \lim_{x \to \infty} (-x)$

$= \infty - \infty = 0$

???? ← Totally meaningless "Infinity" is not a number.

<u>Correct</u>: $x^2 - x = x(x-1)$

$\lim_{x \to \infty} x = \infty$

<u>and</u>

$\lim_{x \to \infty} x - 1 = \infty$

F/ $\Rightarrow \lim_{x \to \infty} x(x-1) = \infty$

2/ $\lim_{x \to 0^-} e^{1/x}$

$\lim_{x \to 0^-} 1/x = -\infty$ G/ $\Rightarrow \lim_{x \to 0^-} e^{1/x} = \lim_{u \to -\infty} e^u = 0$