Limits of Functions

Limits are about understanding $f(x)$ as x approaches a specific value.

$f(x)$ = function defined near $x=a$, except possibly not at $x=a$.

Examples

$f(x)=x$, $a=2$; $f(x)=\frac{1}{x}$, $a=0$;

$f(x)=\frac{x^2-1}{x-1}$, $a=1$.

Intuitive Definitions

approaches

$\lim_{x\to a} f(x) = L$ or $f(x)\to L$ as $x\to a$

Limit

$\iff$ $f(x)$ approaches L as x approaches (but does not equal) a

approaches from above

$\lim_{x\to a^+} f(x) = L$ or $f(x)\to L$ as $x\to a^+$

$\iff$ $f(x)$ approaches L as x approaches (but does not equal) a from above

approaches from below

$\lim_{x\to a^-} f(x) = L$ or $f(x)\to L$ as $x\to a^-$

$\iff$ $f(x)$ approaches L as x approaches (but does not equal) a from below

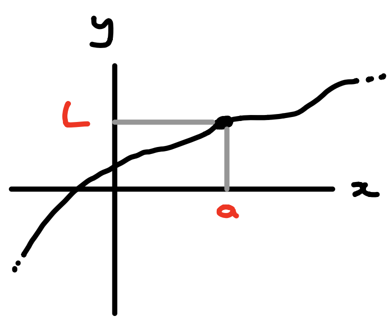
Remarks

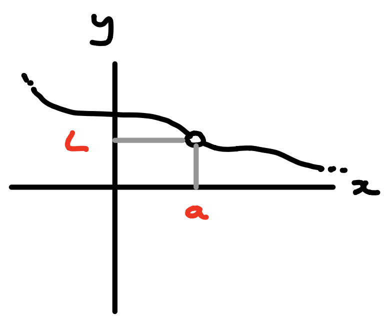
1) Limits don't care about $f(a)$.

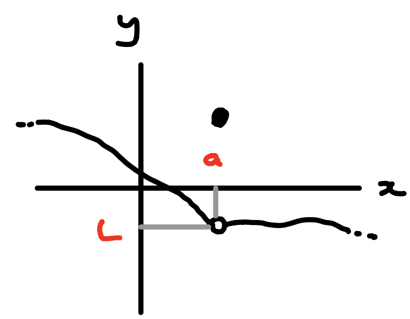
2) If there is no such L we say "the limit does not exist". ← DNE

Basic Graphs

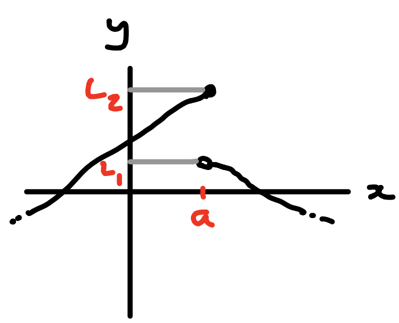
$$\lim_{x \to a} f(x) = L$$

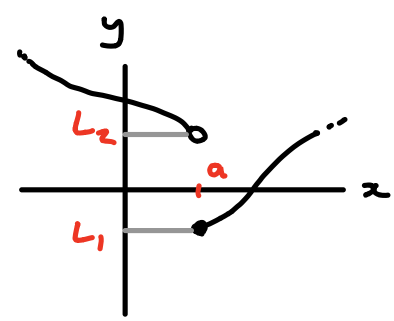


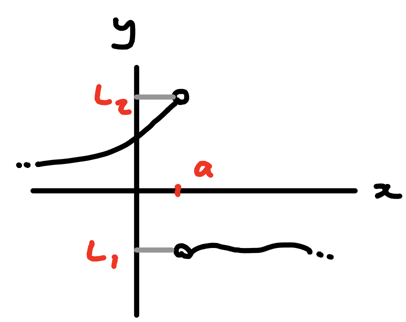




$\lim_{x \to a^+} f(x) = L_1$, $\lim_{x \to a^-} f(x) = L_2$, $\lim_{x \to a} f(x)$ DNE





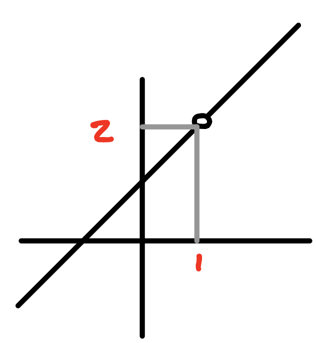


Algebraic Example

$$f(x) = \frac{x^2 - 1}{x - 1} \quad , \quad a = 1$$

$$x \neq 1 \Rightarrow f(x) = \frac{(x+1)(x-1)}{(x-1)} = x + 1$$

$$x = 1 \Rightarrow \text{undefined}$$



$$\Rightarrow \lim_{x \to 1} \frac{x^2-1}{x-1} = 2$$

Remark $\lim_{x \to a} f(x) = L \iff \lim_{x \to a^+} f(x) = L$ and $\lim_{x \to a^-} f(x) = L$

Infinite Limits

Not a number
Just notation

$\lim_{x \to a} f(x) = \infty$ or $f(x) \to \infty$ as $x \to a$ $\iff$

$f(x)$ grows positively without bound as x approaches (but does not equal) a

$\lim_{x \to a} f(x) = -\infty$ or $f(x) \to -\infty$ as $x \to a$ $\iff$

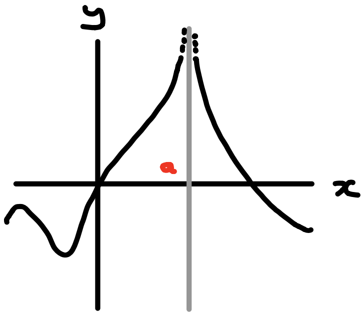
$f(x)$ grows negatively without bound as x approaches (but does not equal) a

We have similar definitions for $\lim_{x \to a^+} f(x) = \infty$, $\lim_{x \to a^+} f(x) = -\infty$, $\lim_{x \to a^-} f(x) = \infty$, $\lim_{x \to a^-} f(x) = -\infty$.

Important : In all cases the limit DNE. It just fails to exist in a specific way.

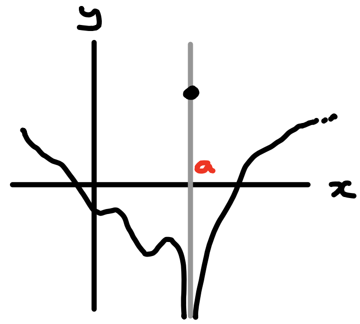
By abuse of terminology we say limit is plus/minus infinity

Basic Graphs



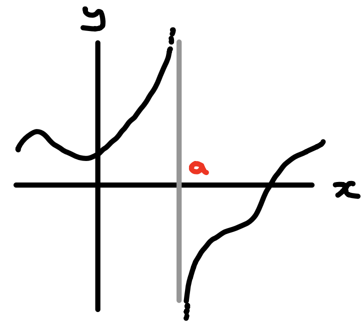
$\lim_{x \to a} f(x) = \infty$

$f(a)$ undefined



$\lim_{x \to a} f(x) = -\infty$

$f(a)$ defined



$\lim_{x \to a^-} f(x) = \infty$

$\lim_{x \to a^+} f(x) = -\infty$

$\lim_{x \to a} f(x)$ DNE and is neither $\pm\infty$

Remark

1) If any of these occur we say $x = a$ is a <u>vertical asymptote</u>.

Examples

$\lim_{x \to 0^+} \frac{1}{x} = \infty$, $\lim_{x \to 0^-} \frac{1}{x} = -\infty$

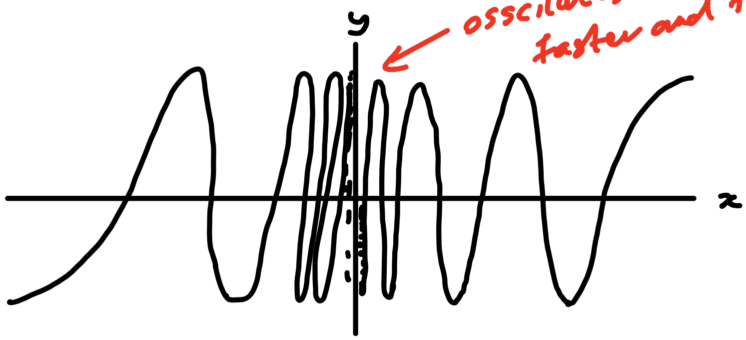
$\lim_{x \to \frac{\pi}{2}^-} \tan(x) = \infty$, $\lim_{x \to -\frac{\pi}{2}^+} \tan(x) = -\infty$

$\lim_{x \to 0^+} \ln(x) = -\infty$, $\lim_{x \to 0} \frac{1}{x^2} = \infty$

Nasty Example

$f(x) = \sin\left(\frac{1}{x}\right) \quad , \quad x = 0$

osscilates faster and faster



$\lim_{x \to 0} \sin\left(\frac{1}{x}\right)$ DNE

$\lim_{x \to 0} \sin\left(\frac{1}{x}\right) \neq \pm\infty$