L'Hospital's Rule

Suppose f and g are differentiable and $g'(x) \neq 0$ on an open interval I containing a (except possibly at a).

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0 / \infty / -\infty \Rightarrow \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Not obvious

Remark

Also holds for $x \to a^+ / a^- / \infty / -\infty$

Examples

Conditions are met

1/ $$\lim_{x \to 0} \frac{\sin(x)}{x} = \lim_{x \to 0} \frac{\cos(x)}{1} = \cos(0) = 1$$

2/ $$\lim_{x \to \infty} \frac{\ln(x)}{x^2+1} = \lim_{x \to \infty} \frac{\left(\frac{1}{x}\right)}{2x} = \lim_{x \to \infty} \frac{1}{2x^2} = 0$$

3/ $$\lim_{x \to 0^+} x \ln(x) = \lim_{x \to 0^+} \frac{\ln(x)}{(1/x)} = \lim_{x \to 0^+} \frac{\left(\frac{1}{x}\right)}{\left(\frac{-1}{x^2}\right)}$$

$$= \lim_{x \to 0^+} -x = 0$$

More generally :

Indeterminant form

$$\lim_{x \to ?} f(x) = 0 , \lim_{x \to ?} g(x) = \infty \Rightarrow \lim_{x \to ?} f(x) g(x) = ???$$

$$f(x) g(x) = \frac{f(x)}{(1/g(x))} \longleftarrow \text{Apply L'Hospital}$$

4/ $\lim_{x\to\infty} e^x - x = ?$

Indeterminant Form

$\lim_{x\to\infty} e^x = \infty$, $\lim_{x\to\infty} x = \infty \Rightarrow \lim_{x\to\infty} e^x - x = ???$

$\lim_{x\to\infty} e^x - x = \lim_{x\to\infty} x\left(\frac{e^x}{x} - 1\right)$

L'Hospital

$\lim_{x\to\infty} \frac{e^x}{x} = \lim_{x\to\infty} \frac{e^x}{1} = \infty$

$\Rightarrow \lim_{x\to\infty} x\left(\frac{e^x}{x} - 1\right) = \infty$

$\lim_{x\to\infty} x = \infty$

More generally :

Indeterminant Form

$\lim_{x\to ?} f(x) = \infty$, $\lim_{x\to ?} g(x) = \infty \Rightarrow \lim_{x\to ?} f(x) - g(x) = ???$

Convert to Product $g(x)\cdot\left(\frac{f(x)}{g(x)} - 1\right)$ and calculate

$\lim_{x\to ?} \frac{f(x)}{g(x)}$ using L'Hopital. If product is indeterminant

(ie $\lim_{x\to ?} \frac{f(x)}{g(x)} = 1$) apply indeterminant product method.

5/ $\lim_{x\to 0^+} x^x = ?$

(Recall $x = e^{\ln(x)}$)

exponentials are cts.

$\lim_{x\to 0^+} x^x = \lim_{x\to 0^+} e^{x\ln(x)} = e^{\left(\lim_{x\to 0^+} x\ln(x)\right)}$

3/ $\Rightarrow \lim_{x\to 0^+} x\ln(x) = 0 \Rightarrow \lim_{x\to 0^+} x^x = e^0 = 1$

More generally,

$$\lim_{x \to ?} f(x) = 0^+, \quad \lim_{x \to ?} g(x) = 0$$

$$\lim_{x \to ?} f(x) = \infty, \quad \lim_{x \to ?} g(x) = 0 \quad \Rightarrow \quad \lim_{x \to ?} (f(x))^{g(x)} = ???$$

$$\lim_{x \to ?} f(x) = 1, \quad \lim_{x \to ?} g(x) = \infty$$

Indeterminant Form

$$f(x)^{g(x)} = e^{g(x)\ln(f(x))} \Rightarrow \lim_{x \to ?} f(x)^{g(x)} = e^{\lim_{x \to ?} g(x)\ln(f(x))}$$

Calculate using indeterminant product method.