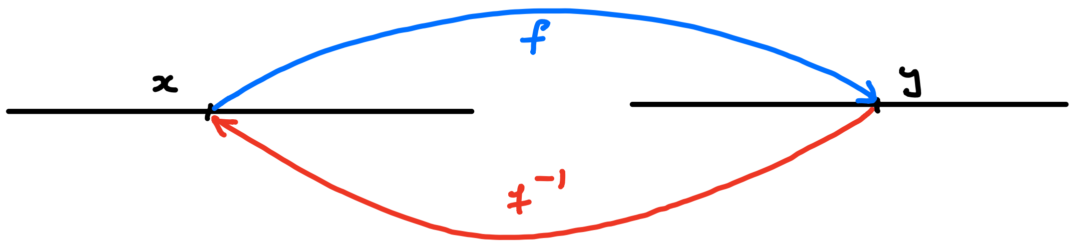
Inverse Functions

Q: When does a function, f, have an inverse, f^{-1}?

Notation! Not $\frac{1}{f}$.

What f^{-1} should do:

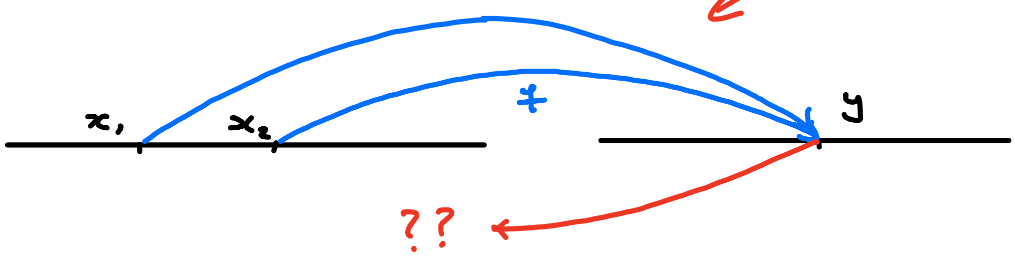


ie for all x in domain of f,

$$f(x) = y \iff f^{-1}(y) = x$$

Potential Problem:

E.g. $(-1)^2 = 1^2 = 1$



$\Rightarrow$ There can be no well-defined f^{-1}.

Definition

Also called injective

We say f is <u>one-to-one</u> if, given x_1, x_2 in domain,

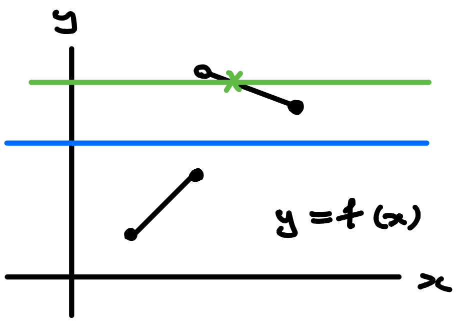
$$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$$

Observation

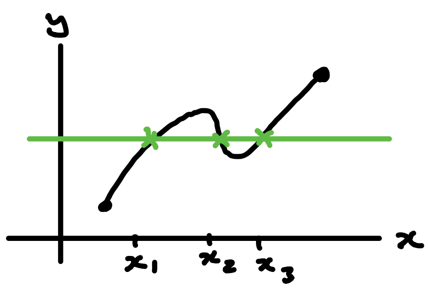
f one-to-one $\iff$ Any horizontal line crosses $y = f(x)$ at most once

could be never

Examples



$\Rightarrow$ f one-to-one



$\Rightarrow$ f not one-to-one

$(f(x_1) = f(x_2) = f(x_3))$

Definition Let f be a one-to-one function with domain A and range B. The inverse of f, denoted f^{-1}, is the function with domain B and range A such that

$$f^{-1}(y) = x \iff f(x) = y \quad \text{for all } y \text{ in } B.$$

Cancellation Properties:

1/ $(f^{-1} \circ f)(x) = x$ for all x in A

2/ $(f \circ f^{-1})(y) = y$ for all y in B

Calculating Inverses

Assuming f one-to-one

Algebraically

with x as independent variable in f^{-1}

Set $f(y) = x$ and solve for y.

$\Rightarrow y = f^{-1}(x)$

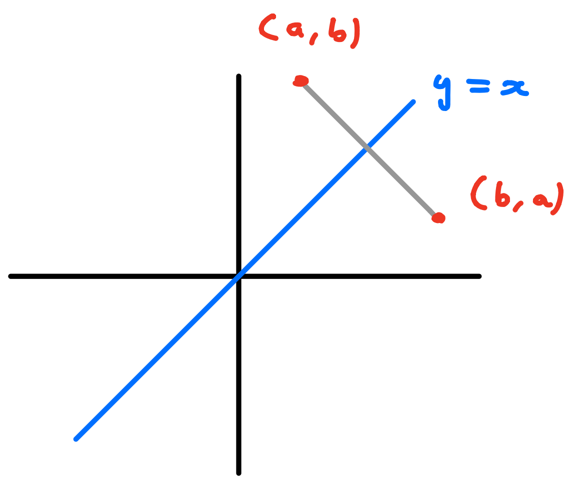
Example $f(x) = 2(x+3)^3$

$$f(y) = x \Rightarrow 2(y+3)^3 = x \Rightarrow y = \sqrt[3]{\frac{x}{2}} - 3$$

Visually

$$y = f(x) \Leftrightarrow x = f^{-1}(y)$$

$\Rightarrow$ Graph of $y = f^{-1}(x)$ is the graph $y = f(x)$ with x and y coordinates interchanged.



$\Rightarrow$ $y = f^{-1}(x)$ is the graph $y = f(x)$ reflected in $y = x$.

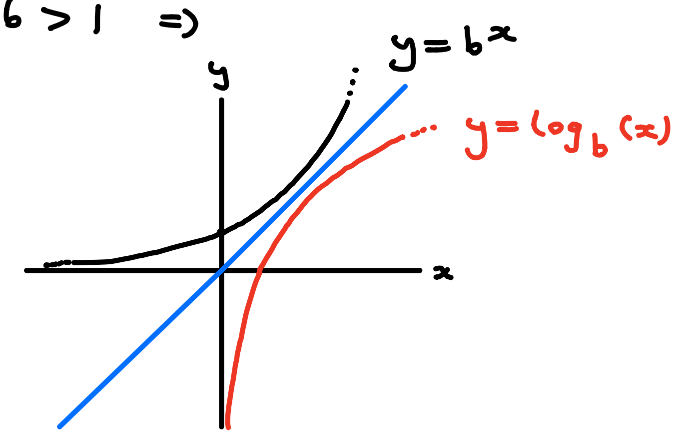
Logarithmic Functions

$b > 0$

$f(x) = b^x$ increasing / decreasing on $\mathbb{R}$ $\Rightarrow$ one-to-one

$f^{-1}(x) = \log_b(x)$ $\leftarrow$ Logarithm function with base b

$b > 1 \Rightarrow$



Domain of $\log_b = (0, \infty)$

$\Rightarrow$ Range of $\log_b = \mathbb{R}$

Cancellation Properties :

1/ $\log_b(b^x) = x$

2/ $b^{\log_b(x)} = x$

Laws of Exponents $\Rightarrow$ Laws of Logarithms

1/ $\log_b(xy) = \log_b(x) + \log_b(y)$

2/ $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$ $\leftarrow$ $x, y > 0$

3/ $\log_b(x^r) = r\log_b(x)$ — r in $\mathbb{R}$

LN

Most Important Example : $\log_e(x) = \ln(x)$

Observation :

Raise both sides to power $\log_b(x)$, $x > 0$

$$e^{\ln(b)} = b$$

$$\Rightarrow e^{\underline{\ln(b)\log_b(x)}} = b^{\log_b(x)} = x = e^{\underline{\ln(x)}}$$

$$\Rightarrow \ln(b)\log_b(x) = \ln(x) \Rightarrow \boxed{\log_b(x) = \frac{\ln(x)}{\ln(b)}}$$

Example Solve $2^{(3x+7)} = 4$

$$\Rightarrow \ln\left(2^{(3x+7)}\right) = \ln(4)$$

$$\Rightarrow (3x+7)\ln(2) = \ln(2^2) = 2\ln(2)$$

$$\Rightarrow 3x+7 = 2 \Rightarrow x = \frac{-5}{3}$$

Inverse Trigonometric Functions

If f not one-to-one we can potentially restrict domain to make it one-to-one.

Example $f(x) = x^2$ on $[0,\infty)$, $f^{-1}(x) = \sqrt{x}$

$\sin(x)$, $\cos(x)$ and $\tan(x)$ are not one-to-one

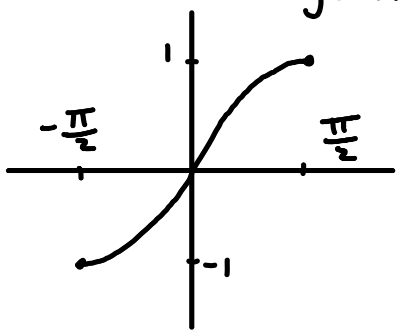
Restrict each domain as follows :

$\sin(x)$ to $\left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$

$\cos(x)$ to $[0,\pi]$

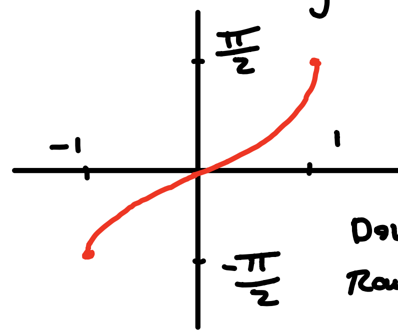
$\tan(x)$ to $\left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$

$y = \sin(x)$



$\Rightarrow$

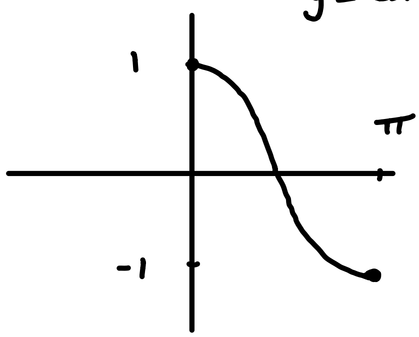
$y = \sin^{-1}(x) = \arcsin(x)$



Domain $= [-1, 1]$

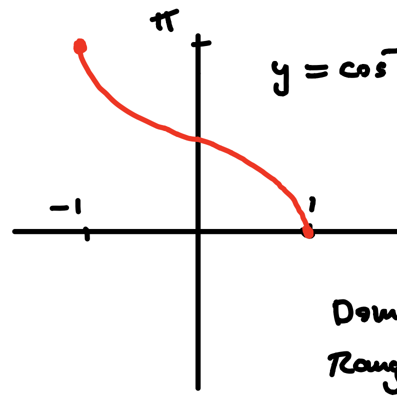
Range $= [-\frac{\pi}{2}, \frac{\pi}{2}]$

$y = \cos(x)$



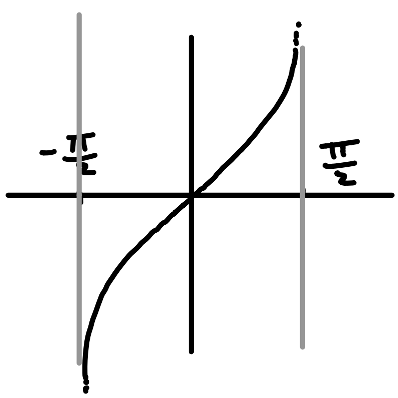
$\Rightarrow$

$y = \cos^{-1}(x) = \arccos(x)$



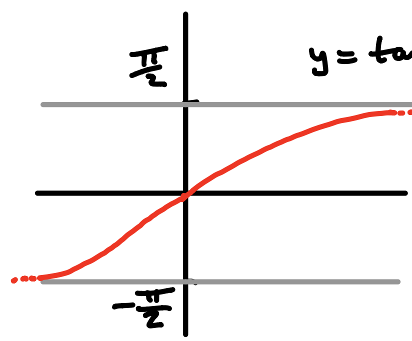
Domain $= [-1, 1]$

Range $= [0, \pi]$



$\Rightarrow$

$y = \tan^{-1}(x) = \arctan(x)$



Domain $= \mathbb{R}$

Range $= (-\frac{\pi}{2}, \frac{\pi}{2})$