Inverse Functions

Q: When does a function, f, have an inverse, \( f^{-1} \)?

What \( f^{-1} \) should do:

\[ x \overset{f}{\rightarrow} y \overset{f^{-1}}{\rightarrow} x \]

i.e. for all \( x \) in domain of \( f \),

\[ f(x) = y \iff f^{-1}(y) = x \]

Potential Problem:

\[ x, x', x'' \overset{f}{\rightarrow} y \]

\[ ?? \]

\[ \Rightarrow \text{There can be no well-defined } f^{-1}. \]

Definition

We say \( f \) is one-to-one it, given \( x_1, x_2 \) in domain,

\[ x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2) \]

Also called injective.
Observation

\[ \text{if one-to-one } \iff \text{Any horizontal line crosses } y = f(x) \text{ at most once} \]

\[ \rightarrow \text{ could be never} \]

Examples

\[ \text{\( y = f(x) \)} \]

\[ \Rightarrow \text{ not one-to-one} \]

\[ (f(x_1) = f(x_2) = f(x_3)) \]

Definition Let \( f \) be a one-to-one function with domain \( A \) and range \( B \). The inverse of \( f \), denoted \( f^{-1} \), is the function with domain \( B \) and range \( A \) such that

\[ f^{-1}(y) = x \iff f(x) = y \text{ for all } y \text{ in } B. \]

Cancellation Properties: \( (f^{-1} \circ f)(x) = x \text{ for all } x \in A \)

\[ (f \circ f^{-1})(y) = y \text{ for all } y \in B. \]
Calculating Inverses

Assuming $f$ one-to-one

Algebraically

with $x$ as independent variable in $f^{-1}$

Set $f(y) = x$ and solve for $y$.

$y = f^{-1}(x)$

Example $f(x) = 2(x + 3)^3$

$f(y) = x \implies 2(y + 3)^3 = x \implies y = \sqrt[3]{\frac{x}{2}} - 3$

Visually

$y = f(x) \iff x = f^{-1}(y)$

$\implies$ Graph of $y = f^{-1}(x)$ is the graph $y = f(x)$

with $x$ and $y$ coordinates interchanged.

$y = f^{-1}(x)$ is reflected in $y = x$. 
Logarithmic Functions

$b > 0$

$y(x) = b^x$ increasing/decreasing on $\mathbb{R} \Rightarrow$ one-to-one

$y^{-1}(x) = \log_b(x)$ \(\text{Logarithm function with base } b\)

\[ b > 1 \Rightarrow \quad y = b^x \]

\[ \text{Domain of } \log_b = (0, \infty) \]

\[ \Rightarrow \quad \text{Range of } \log_b = \mathbb{R} \]

Cancellation Properties:

1. $\log_b(b^x) = x$
2. $b^{\log_b(x)} = x$

Laws of Exponents $\Rightarrow$ Laws of Logarithms

1. $\log_b(xy) = \log_b(x) + \log_b(y)$
2. $\log_b\left(\frac{x}{y}\right) = \log_b(x) - \log_b(y)$ \(x, y > 0\)
3. $\log_b(x^r) = r \log_b(x)$ \(r \in \mathbb{R}\)

Most Important Example: $\log_e(x) = \ln(x)$
Observation:

\[ e^{\ln(b)} = b \]

Raise both sides to power \( \log_b(x) \), \( x > 0 \)

\[ e^{\ln(b) \log_b(x)} = \log_b(x) = x = e^{\ln(x)} \]

\[ \ln(b) \log_b(x) = \ln(x) \Rightarrow \log_b(x) = \frac{\ln(x)}{\ln(b)} \]

Example: Solve \( 2^{(3x+7)} = 4 \)

\[ \ln(2^{(3x+7)}) = \ln(4) \]

\[ (3x+7) \ln(2) = 2 \ln(2) = 2 \ln(2) \]

\[ 3x+7 = 2 \Rightarrow x = -\frac{5}{3} \]

Inverse Trigonometric Functions

If \( f \) is not one-to-one we can potentially restrict domain to make it one-to-one.

Example: \( f(x) = x^2 \) on \( [0, \infty) \), \( f^{-1}(x) = \sqrt{x} \)

\( \sin(x), \cos(x) \) and \( \tan(x) \) are not one-to-one

Restrict each domain as follows:

- \( \sin(x) \) to \( \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right] \)
- \( \cos(x) \) to \( [0, \pi] \)
- \( \tan(x) \) to \( \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \)
\[ y = \sin(x) \]

**Domain:** \([-\pi, \pi]\)

**Range:** \([-1, 1]\)

\[ y = \cos(x) \]

**Domain:** \([-\pi, \pi]\)

**Range:** \([-1, 1]\)

\[ y = \tan(x) \]

**Domain:** \(\mathbb{R} \setminus \{\pm \pi/2\} \)

**Range:** \((\pm\pi, \pm\pi)\)

\[ y = \sin^{-1}(x) = \arcsin(x) \]

**Domain:** \([-1, 1]\)

**Range:** \([-\pi/2, \pi/2]\)

\[ y = \cos^{-1}(x) = \arccos(x) \]

**Domain:** \([-1, 1]\)

**Range:** \([0, \pi]\)

\[ y = \tan^{-1}(x) = \arctan(x) \]

**Domain:** \(\mathbb{R} \)

**Range:** \((\pm\pi/2, \pm\pi/2)\)