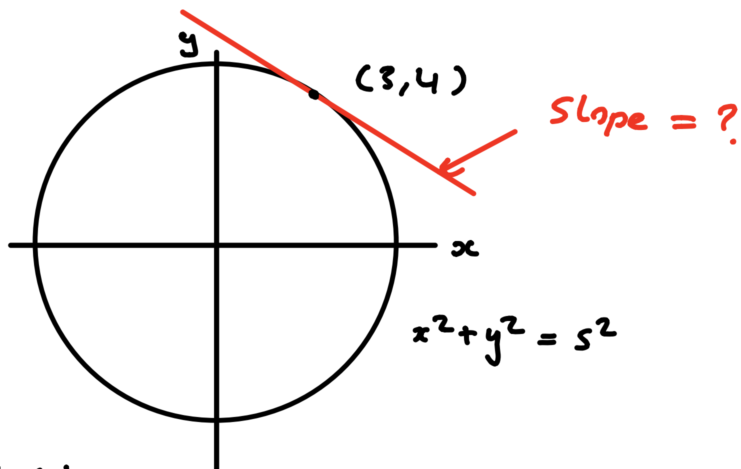
Implicit Differentiation

Q: What is the slope of the tangent line to $x^2+y^2=5^2$ at $(3,4)$?



Approach 1 (Explicit)

Solve $x^2+y^2=5^2$ in y and differentiate.

$x^2+y^2=5^2 \Rightarrow \quad y = \pm\sqrt{5^2-x^2}$

$(3,4)$ in top half-circle, hence we should choose

$y=\sqrt{5^2-x^2}$.

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{2\sqrt{5^2-x^2}} \cdot (-2x)$$

$$\Rightarrow \quad \left.\frac{dy}{dx}\right|_{x=3} = \frac{1}{2\sqrt{5^2-3^2}} \cdot (-2\cdot 3) = \frac{-6}{2\cdot 4} = \frac{-3}{4}$$

__Potential Problem__ : May be impossible to explicity solve in y.

__Approach 2 (Implicit)__

Without explicitly solving for y, assume $y = f(x)$ for some function f (at least near to $(3,4)$). $f(3) = 4$

$$\Rightarrow \quad x^2 + (f(x))^2 = 5^2$$ equality of functions

$$\Rightarrow \quad \frac{d}{dx}\left(x^2 + (f(x))^2\right) = \frac{d}{dx}(5^2)$$ constant function

Chain Rule

$$\Rightarrow \quad 2x + 2(f(x))f'(x) = 0$$

$$\Rightarrow \quad f'(x) = \frac{-2x}{2f(x)} = \frac{-x}{f(x)}$$

$$\Rightarrow \quad f'(3) = \frac{-3}{f(3)} = \frac{-3}{4}$$

__Remark__

1/ This approach works well if we cannot solve in y

2/ Its generally better to do this in Liebniz notation:

$$x^2+y^2=5^2 \Rightarrow \frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(5^2)$$

$$\Rightarrow \quad 2x + \frac{dy^2}{dy} \cdot \frac{dy}{dx} = 0$$ Chain Rule

$$\Rightarrow \quad 2x + 2y \cdot \frac{dy}{dx} = 0 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{-2x}{2y} = \frac{-x}{y}$$

The only drawback is that $\frac{dy}{dx}$ will be in terms of both x and y.

Example

1/ $\quad y^2 + 2y + 5 = x \quad \Rightarrow \quad \frac{dy}{dx} = ?$

$$\Rightarrow \quad \frac{d}{dx}\left(y^2 + 2y + 5\right) = \frac{d}{dx}(x)$$

$$\Rightarrow \quad \frac{dy^2}{dy} \cdot \frac{dy}{dx} + \frac{d2y}{dy} \cdot \frac{dy}{dx} - \frac{d}{dx}(5) = 1$$

$$\Rightarrow \quad (2y + 2)\frac{dy}{dx} = 1 \quad \Rightarrow \quad \frac{dy}{dx} = \frac{1}{2y+2}$$

2/ $\quad \sin(xy) = y \quad \Rightarrow \quad \frac{dy}{dx} = ?$

$$\Rightarrow \quad \frac{d}{dx}\left(\sin(xy^3)\right) = \frac{dy}{dx}$$

Chain Rule

$$\Rightarrow \quad \cos(xy^3)\frac{d}{dx}(xy^3) = \frac{dy}{dx}$$

$$\Rightarrow \quad \cos(xy^3)\left(\frac{d}{dx}(x) \cdot y^3 + x \cdot \frac{d}{dx}(y^3)\right) = \frac{dy}{dx}$$

Chain Rule

$$\Rightarrow \quad \cos(xy^3)\left(y^3 + x \cdot 3y^2 \cdot \frac{dy}{dx}\right) = \frac{dy}{dx}$$

$$\Rightarrow \quad \cos(xy^3) \cdot y^3 + \cos(xy^3) \cdot 3 \cdot xy^2 \frac{dy}{dx} = \frac{dy}{dx}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{\cos(xy^3)\, y^3}{1 - \cos(xy^3) \cdot 3xy^2}$$

Overview of Implicit Differentiation

1) Differentiate both sides of equation with respect to x
2) Using laws of differentiation (Product, Chain, Sum ...) expand both sides as far as possible.
3) Solve in $\frac{dy}{dx}$.

Differentiating Inverse Functions

definition of inverse function

Recall : $y = f^{-1}(x) \iff f(y) = x$

Q : What is $\frac{d}{dx}(f^{-1}(x)) = \frac{dy}{dx}$?

$f(y) = x$

$$\Rightarrow \frac{d}{dx}(f(y)) = \frac{d}{dx}(x) \Rightarrow \frac{d}{dy}(f(y)) \cdot \frac{dy}{dx} = 1$$

$$\Rightarrow f'(y) \cdot \frac{dy}{dx} = 1 \Rightarrow \frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{f'(f^{-1}(x))}$$

Conclusion : $$\boxed{\frac{d}{dx}(f^{-1}(x)) = \frac{1}{f'(f^{-1}(x))}}$$

Examples

1) $y = \arcsin(x) \iff \sin(y) = x$ $\quad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$$\Rightarrow \frac{d}{dx}(\sin(y)) = \frac{d}{dx}(x)$$

$\cos(y) \geq 0$ as $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

$$\Rightarrow \quad \cos(y)\frac{dy}{dx} = 1$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1-\sin^2(y)}} = \frac{1}{\sqrt{1-x^2}}$$

2/ $y = \arccos(x) \iff \cos(y) = x$ $\quad 0 \leq y \leq \pi$

$$\Rightarrow \quad \frac{d}{dx}(\cos(y)) = \frac{d}{dx}(x)$$

$\sin(y) \geq 0$ as $0 \leq y \leq \pi$

$$\Rightarrow \quad -\sin(y)\frac{dy}{dx} = 1$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{-\sin(y)} = \frac{1}{-\sqrt{1-\cos^2(y)}} = \frac{-1}{\sqrt{1-x^2}}$$

3/ $y = \arctan(x) \iff \tan(y) = x$ $\quad -\frac{\pi}{2} < y < \frac{\pi}{2}$

$$\Rightarrow \quad \frac{d}{dx}(\tan(y)) = \frac{d}{dx}(x)$$

$\cos^2(y) + \sin^2(y) = 1$ Divide by $\cos^2(y)$

$\Rightarrow 1 + \tan^2(y) = \sec^2(y)$

$$\Rightarrow \quad \sec^2(y)\frac{dy}{dx} = 1$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{1}{\sec^2(y)} = \frac{1}{1+\tan^2(y)} = \frac{1}{1+x^2}$$