Final Review

Indefinite integral

$$\int f(x)\,dx = \text{General antiderivative of } f$$

$$= F(x) + C$$

on each disjoint open interval in domain

$F'(x) = f(x)$

$$F(x) \xrightarrow{\frac{d}{dx}} f(x) \xrightarrow{\frac{d}{dx}} f'(x)$$

Core Examples / Rules

$r \neq -1$

$$\int x^r\,dx = \frac{1}{r+1}x^{r+1} + C$$

$$\int x^{-1}\,dx = \ln|x| + C$$

$b \neq 1$

$$\int b^x\,dx = \frac{1}{\ln(b)}b^x + C$$

$$\int \sin(x)\,dx = -\cos(x) + C \quad / \quad \int \cos(x)\,dx = \sin(x) + C$$

$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin(x) + C \quad / \quad \int \frac{1}{1+x^2}\,dx = \arctan(x) + C$$

$$\int f(x) + g(x)\,dx = \int f(x)\,dx + \int g(x)\,dx$$

$$\int c f(x)\,dx = c\int f(x)\,dx$$

Example

$$\int \sqrt{x} + \sin(x)\,dx = \int \sqrt{x}\,dx + \int \sin(x)\,dx$$

$$= \int x^{\frac{1}{2}}\, dx + \int \sin(x)\, dx$$

$$= \frac{1}{\frac{1}{2}+1} x^{\frac{1}{2}+1} + (-\cos(x)) + C$$

$$= \frac{2}{3} x^{3/2} - \cos(x) + C$$

Chain Rule:

$$\frac{d}{dx}\left(F(g(x))\right) = F'(g(x))\, g'(x) = f(g(x))\, g'(x)$$

$$\Rightarrow \quad \int \underline{\underline{f(g(x))\, g'(x)}}\, dx = F(g(x)) + C$$

Common Examples

$$\int f(\underline{x^k}) \cdot \underline{k\, x^{k-1}}\, dx, \quad \int f(\underline{b}^x)\, \underline{\ln(b)\, b^x}\, dx$$

$$\int f(\underline{\ln(x)}) \cdot \frac{1}{\underline{x}}\, dx, \quad \int f(\underline{\sin(x)})\, \underline{\cos(x)}\, dx,$$

$$\int f(\underline{\cos(x)})\, (\underline{-\sin(x)})\, dx, \quad \int f(\underline{\arcsin(x)}) \cdot \frac{1}{\underline{\sqrt{1-x^2}}}\, dx,$$

$$\int f(\underline{\arctan(x)}) \cdot \frac{1}{\underline{1+x^2}}\, dx$$

Remark It's generally easier to spot $g(x)$ than $f(x)$.

Example $f(g(x))\, g'(x)$

1/ $x \sin(x^2) = \frac{1}{2} \sin(x^2) \cdot 2x \Rightarrow$ $f(x) = \frac{1}{2}\sin(x)$, $g(x) = x^2$

2/ $x^3\sqrt{x^2+1} = \frac{1}{2}x^2\sqrt{x^2+1}\cdot 2x$

$$= \underbrace{\frac{1}{2}\left((x^2+1)-1\right)\sqrt{x^2+1}}_{f(x^2+1)} \cdot \underbrace{2x}_{2x}$$

$\Rightarrow f(x) = \frac{1}{2}(x-1)\sqrt{x} = \frac{1}{2}x^{3/2} - \frac{1}{2}x^{1/2}$

$g(x) = x^2+1$

Integration by Substitution

Chain Rule in reverse

x-world

$$\left(\begin{array}{l} u = g(x) \\ \frac{du}{dx} = g'(x) \\ dx = \frac{du}{g'(x)} \end{array}\right)$$

u-world

Integrate in u-world

$$\int f(g(x))g'(x)\,dx = F(g(x))+C = F(u)+C = \int f(u)\,du$$

Replace dx with expression in du

Replace u with g(x)

Replace g(x) with u

$$\int f(g(x))\cancel{g'(x)}\frac{du}{\cancel{g'(x)}} = \int f(g(x))\,du$$

Example $\int x^3\sqrt{x^2+1}\,dx = ?$

Let $u = x^2+1 \Rightarrow \frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x}$

$x^2 = u-1$

$\Rightarrow \int x^3\sqrt{x^2+1}\,dx = \int x^3\sqrt{x^2+1}\,\frac{du}{2x}$

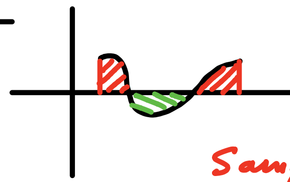
$$= \int \frac{1}{2} x^2 \sqrt{x^2+1}\, du = \int \frac{1}{2} (u-1)\sqrt{u}\, du$$

$$= \int \frac{1}{2} u^{3/2} - \frac{1}{2} u^{1/2}\, du = \frac{1}{2} \frac{1}{1+\frac{3}{2}} u^{\frac{3}{2}+1} - \frac{1}{2} \frac{1}{\frac{1}{2}+1} u^{\frac{1}{2}+1} + C$$

$$= \frac{1}{5} u^{5/2} - \frac{1}{3} u^{3/2} + C = \frac{1}{5} (x^2+1)^{5/2} - \frac{1}{3} (x^2+1)^{3/2} + C$$

Definite Integral

$\int_a^b f(x)\,dx$ Intuitive



$$\Rightarrow \int_a^b f(x)\,dx = \text{Area}(\text{red}) - \text{Area}(\text{green})$$

Net area

$\int_a^b f(x)\,dx$ Precise

Sample point in $[x_{i-1}, x_i]$ e.g. $x_i^* = x_i$; $\frac{b-a}{n}$

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} \underbrace{\sum_{i=1}^{n} f(x_i^*)\,\Delta x}_{\text{Riemann Sum}}$$

Total area

enclosed by $y = f(x)$ and x-axis between $x = a$ and b $= \int_a^b |f(x)|\,dx$

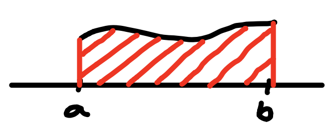
Fundamental Theorem of Calculus

cts on $[a,b]$

$$\int f(x)\,dx = F(x) + C \Rightarrow \int_a^b f(x)\,dx = F(b) - F(a) = F(x)\Big|_a^b$$

Observation

$f(x) \geq 0$ on $[a,b]$ $\Rightarrow$



Total area enclosed by $y=f(x)$ and x-axis between $x=a$ and b

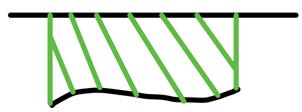
$$= \int_a^b |f(x)|\,dx = \int_a^b f(x)\,dx = \left|\int_a^b f(x)\,dx\right|$$

$f(x) \leq 0$ on $[a,b]$ $\Rightarrow$

Total area enclosed by $y=f(x)$ and x-axis between $x=a$ and b

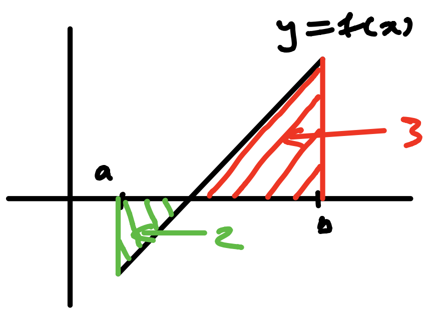
$$= \int_a^b |f(x)|\,dx = \int_a^b -f(x)\,dx = \left|\int_a^b f(x)\,dx\right|$$

$$= -\int_a^b f(x)\,dx$$



Warning : Won't work if $f(x)$ isn't always ≥ 0 or ≤ 0.

E.g.



$$\Rightarrow \int_a^b |f(x)|\,dx = 2+3 = 5$$

$$\neq$$

$$\left|\int_a^b f(x)\,dx\right| = |3-2| = 1$$

Conclusion : Calculating $\int_a^b |f(x)|\,dx$

1/ Do sign analysis on $f(x)$ on $[a,b]$.

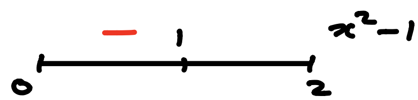
2/ Calculate $\left| \int_c^d f(x)\,dx \right|$ for each subinterval.

3/ Sum to calculate $\int_a^b |f(x)|\,dx$.

<u>Example</u> $\int_0^2 |x^2-1|\,dx$

1/ A/ $x^2-1=0 \iff x=1$ ← $[0,2]$

B/ None



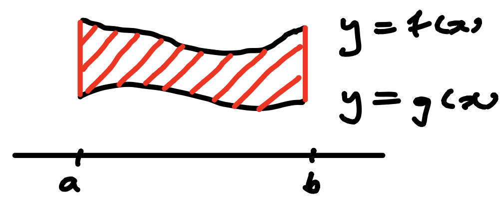
2/ $\left| \int_0^1 x^2-1\,dx \right| = \left| \frac{1}{3}x^3 - x \Big|_0^1 \right| = \left| \frac{-2}{3} \right| = \frac{2}{3}$

$\left| \int_1^2 x^2-1\,dx \right| = \left| \frac{1}{3}x^3 - x \Big|_1^2 \right| = \left| \frac{4}{3} \right| = \frac{4}{3}$

3/ $\int_0^2 |x^2-1|\,dx = \frac{2}{3} + \frac{4}{3} = 2$

<u>Area Between Curves</u>

$f(x) \geq g(x)$ on $[a,b]$



<u>Total</u> Area enclosed by

$y=f(x)$ and $y=g(x)$

between $x=a$ and $x=b$

$= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

$= \int_a^b f(x)-g(x)\,dx$

General Case :

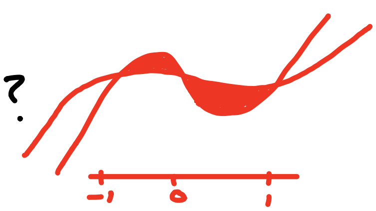
Calculate as above

Total Area enclosed by $y = f(x)$ and $y = g(x)$ between $x = a$ and $x = b$

$$= \int_a^b |f(x) - g(x)|\, dx$$

Example

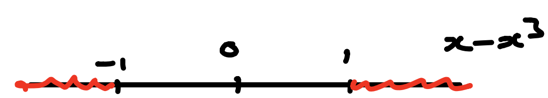
Total Area enclosed by $y = x^3 + x + 1$ and $y = 2x^3 + 1$ $= ?$



$$\underbrace{(x^3 + x + 1)}_{f(x)} - \underbrace{(2x^3 + 1)}_{g(x)} = x - x^3 = x(1-x)(1+x)$$

A/ $x - x^3 = 0 \iff x = -1, 0, 1$

B/ None

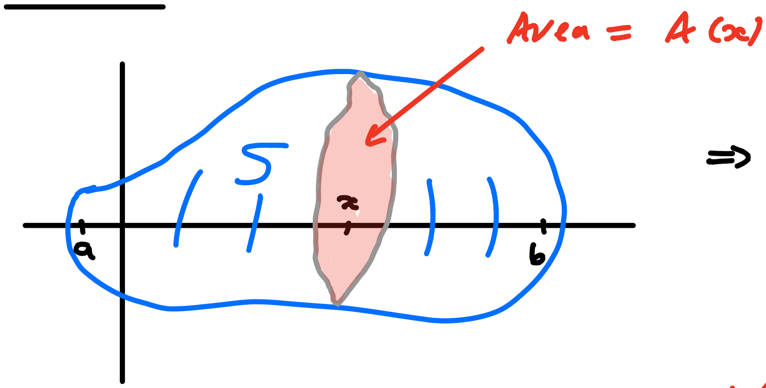


$$\left| \int_{-1}^{0} x - x^3 \, dx \right| = \left| \frac{1}{2}x^2 - \frac{1}{4}x^4 \Big|_{-1}^{0} \right| = \frac{1}{4}$$

$$\left| \int_{0}^{1} x - x^3 \, dx \right| = \left| \frac{1}{2}x^2 - \frac{1}{4}x^4 \Big|_{0}^{1} \right| = \frac{1}{4}$$

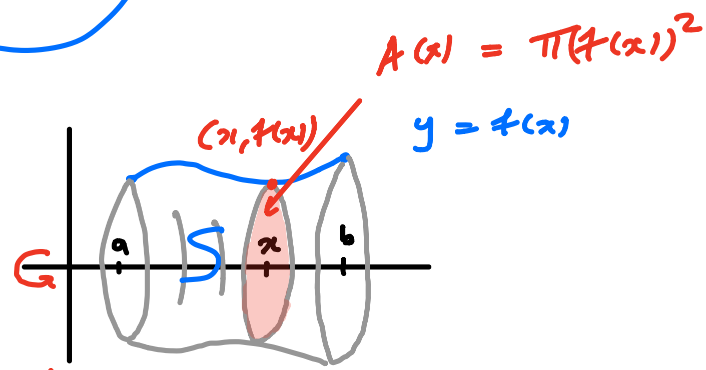
$\Rightarrow$ Total area $= \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$

Volumes



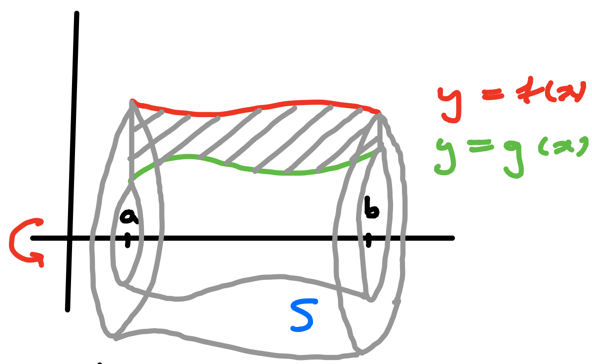
$$\Rightarrow \quad Vol(S) = \int_a^b A(x)\,dx$$

Special Case :



$$\Rightarrow \quad Vol(S) = \int_a^b \underbrace{\pi (f(x))^2}_{A(x)}\,dx$$

$f(x) \geq g(x) \geq 0$ on $[a,b]$



$$Vol(S) = \int_a^b \pi (f(x))^2\,dx - \int_a^b \pi (g(x))^2\,dx$$

$$= \int_a^b \pi \left((f(x))^2 - (g(x))^2 \right) dx$$

$$\neq \pi (f(x) - g(x))^2$$

General Case : $(g(x), f(x) \geq 0$ on $[a,b])$

$\Rightarrow$

Volume of solid of revolution of region enclosed by $y = f(x)$ and $y = g(x)$ between a and b about x-axis

$$= \int_a^b \left| \pi \left(f(x)^2 - g(x)^2 \right) \right| dx$$

Example

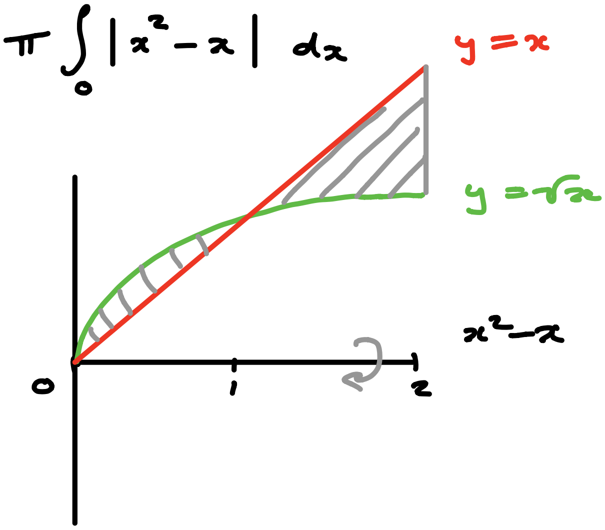
Volume of solid of revolution of region enclosed by $y = \sqrt{x}$ and $y = x$ between 0 and 2 about x-axis

$$= \int_0^2 \left| \pi \left(x^2 - (\sqrt{x})^2 \right) \right| dx$$

$$= \pi \int_0^2 \left| x^2 - x \right| dx$$

A/ $x^2 - x = 0 \iff x = 0, 1$

B/ None



$$\left| \pi \int_0^1 x^2 - x \, dx \right| = \left| \pi \left(\frac{1}{3}x^3 - \frac{1}{2}x^2 \right) \Big|_0^1 \right| = \frac{\pi}{6}$$

$$\left|\pi\int_1^2 x^2 - x \, dx\right| = \left|\pi \left(\frac{1}{3}x^3 - \frac{1}{2}x^2\right)\Big|_1^2\right| = \frac{5}{6}\pi$$

$\Rightarrow$

$$\text{Volume} = \frac{\pi}{6} + \frac{5\pi}{6} = \pi$$