MATH 1A FINAL (PRACTICE 3)
PROFESSOR PAULIN

DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.

CALCULATORS ARE NOT PERMITTED

YOU MAY USE YOUR OWN BLANK PAPER FOR ROUGH WORK

SO AS NOT TO DISTURB OTHER STUDENTS, EVERYONE MUST STAY UNTIL THE EXAM IS COMPLETE

REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT

THIS EXAM WILL BE ELECTRONICALLY SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED. YOU MAY WRITE SOLUTIONS ON THE BLANK PAGE AT THE BACK BUT BE SURE TO CLEARLY LABEL THEM

Name: ______________________________

Student ID: ______________________________

GSI's name: ______________________________

This exam consists of 10 questions. Answer the questions in the spaces provided.

1. Calculate the following. You do not need to simplify your answers.

(a) (10 points)

$$\frac{d}{dx}\frac{\ln(x^2+2))}{\tan(x)}$$

Solution:

$$\frac{d}{dx}\frac{\ln(x^2+2)}{\tan(x)} = \frac{\frac{2x}{x^2+2}\tan(x) - \ln(x^2+2)\cdot\sec^2(x)}{(\tan(x))^2}$$

(b) (15 points)

$$\frac{d}{dx}\cos(x^{(e^x)})$$

Solution:

$$f(x) = x^{(e^x)} \Rightarrow \ln(f(x)) = e^x\ln(x)$$

$$\Rightarrow \frac{d}{dx}(\ln(f(x))) = e^x\ln(x) + e^x\cdot\frac{1}{x}$$

$$\Rightarrow f'(x) = x^{e^x}\cdot\left(e^x\ln(x) + e^x\cdot\frac{1}{x}\right)$$

$$\Rightarrow \frac{d}{dx}\cos(x^{(e^x)}) =$$

$$-\sin(x^{(e^x)})\cdot x^{e^x}\cdot\left(e^x\ln(x) + e^x\cdot\frac{1}{x}\right)$$

2. Calculate the following (you do not need to use the (ϵ, δ)-definition):

(a) (10 points)

$$\lim_{x\to 0} \frac{\tan(2x)}{x}$$

Solution:

L'Hospital

$$\lim_{x\to 0} \frac{\tan(2x)}{x} = \lim_{x\to 0} \frac{2\sec^2(2x)}{1} = 2$$

(b) (15 points)

$$\lim_{x\to -\infty} \sqrt{(9x^2 - x)} + 3x$$

Hint: Rationalize.

Solution:

$$\lim_{x\to -\infty} \sqrt{9x^2 - x} + 3x$$

$$= \lim_{x\to -\infty} \sqrt{9x^2 - x} + 3x \cdot \frac{\sqrt{9x^2-x} - 3x}{\sqrt{9x^2-x} - 3x}$$

$$= \lim_{x\to -\infty} \frac{-x}{\sqrt{9x^2-x} - 3x} = \lim_{x\to -\infty} \frac{-1}{\frac{\sqrt{9x^2-x}}{-\sqrt{x^2}} - 3}$$

$$= \lim_{x\to -\infty} \frac{-1}{-\sqrt{9-\frac{1}{x}} - 3} = \frac{-1}{-3-3} = \frac{1}{6}$$

3. Calculate the following (you do not need to use the Riemann sum definition):

(a) (10 points)

$$\int x^3\sqrt{x^2+1}dx$$

Solution:

$u = x^2+1 \Rightarrow \frac{du}{dx} = 2x \Rightarrow dx = \frac{du}{2x}$

$\Rightarrow \int x^3 \sqrt{x^2+1}\, dx = \frac{1}{2}\int x^2 \sqrt{u}\, du = \frac{1}{2}\int (u-1)\sqrt{u}\, du$

$= \frac{1}{2}\int u^{3/2} - u^{1/2}\, du = \frac{1}{2}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + C$

$= \frac{1}{5}(x^2+1)^{5/2} - \frac{1}{3}(x^2+1)^{3/2} + C$

(b) (10 points)

$$\int_1^2 \frac{e^{1/x^2}}{x^3}dx$$

Solution:

$u = \frac{1}{x^2} \Rightarrow \frac{du}{dx} = \frac{-2}{x^3} \Rightarrow dx = \frac{x^3}{-2}\, du$

$\Rightarrow \int \frac{e^{1/x^2}}{x^3}\, dx = \frac{-1}{2}\int e^u\, du = \frac{-1}{2}e^u + C = \frac{-1}{2}e^{1/x^2} + C$

$\Rightarrow \int_1^2 \frac{e^{1/x^2}}{x^3}\, dx = \frac{-1}{2}e^{1/4} + \frac{1}{2}e$

4. (25 points) The ration of carbon-14 to carbon-12 is an ancient wooden artifact is 60 percent that of living organic matter. How old is the artifact?

   Hint: The half-life of carbon-14 is 5730 years.

   Solution:

$M(t)$ = Mass of carbon-14 t years after death of tree.

$M(t) = Ce^{kt}$ and $M(5730) = \frac{1}{2}M(0)$

$\Rightarrow \quad Ce^{5730k} = \frac{1}{2}C \quad \Rightarrow \quad k = \frac{-\ln(2)}{5730}$

$\Rightarrow \quad M(t) = Ce^{\frac{-\ln(2)}{5730}t}$

$M(t) = 0.6\,M(0) \quad \Rightarrow \quad Ce^{-\frac{\ln(2)}{5730}t} = 0.6\,C$

$\Rightarrow \quad t = \dfrac{\ln(0.6)}{\left(\frac{-\ln(2)}{5730}\right)}$ $\longleftarrow$ How long ago tree died / artifact was made.

5. (25 points) Sketch the following curve. Be sure to indicate asymptotes, local maxima and minima and concavity. Show your working on this page and draw the graph on the next page.

$$y = \frac{x^3}{1+x^2}$$

Solution:

$$f(x) = \frac{x^3}{x^2+1}$$

Domain : $\mathbb{R}$

Odd/Even : Odd $\left(f(-x) = -f(x)\right)$

Vertical asymptotes : None

Behavior at $\pm\infty$:

$$\lim_{x \to \pm\infty} \frac{f(x)}{x} = \lim_{x \to \pm\infty} \frac{x^3}{x^3+x} = 1$$

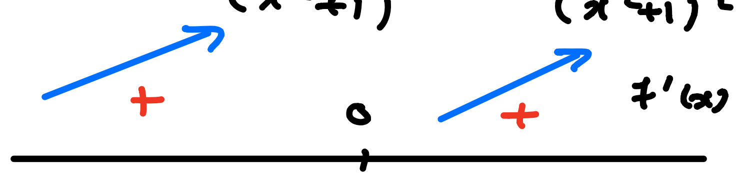
$$\lim_{x \to \pm\infty} f(x) - x = \lim_{x \to \pm\infty} \frac{x^3}{x^2+1} - \frac{x^3+x}{x^2+1} = \lim_{x \to \pm\infty} \frac{-x}{x^2+1} = 0$$

$\Rightarrow$ $y = x$ a slant asymptote

$$f'(x) = \frac{3x^2(x^2+1) - 2x(x^3)}{(x^2+1)^2} = \frac{x^4+3x^2}{(x^2+1)^2} = \frac{x^2(x^2+3)}{(x^2+1)^2}$$

A/ $f'(x) = 0 \iff x = 0$

B/ None



Solution (continued) :

$$f''(x) = \frac{(4x^3+6x)(x^2+1)^2 - 2x\cdot 2(x^2+1)(x^4+3x^2)}{(x^2+1)^4}$$

$$= \frac{(4x^3+6x)(x^2+1) - 4x(x^4+3x^2)}{(x^2+1)^3}$$

$$= \frac{x(6-2x^2)}{(x^2+1)^3}$$

Inflections

$f''(x)$: $+$ $-\sqrt{3}$ $-$ 0 $+$ $\sqrt{3}$ $-$

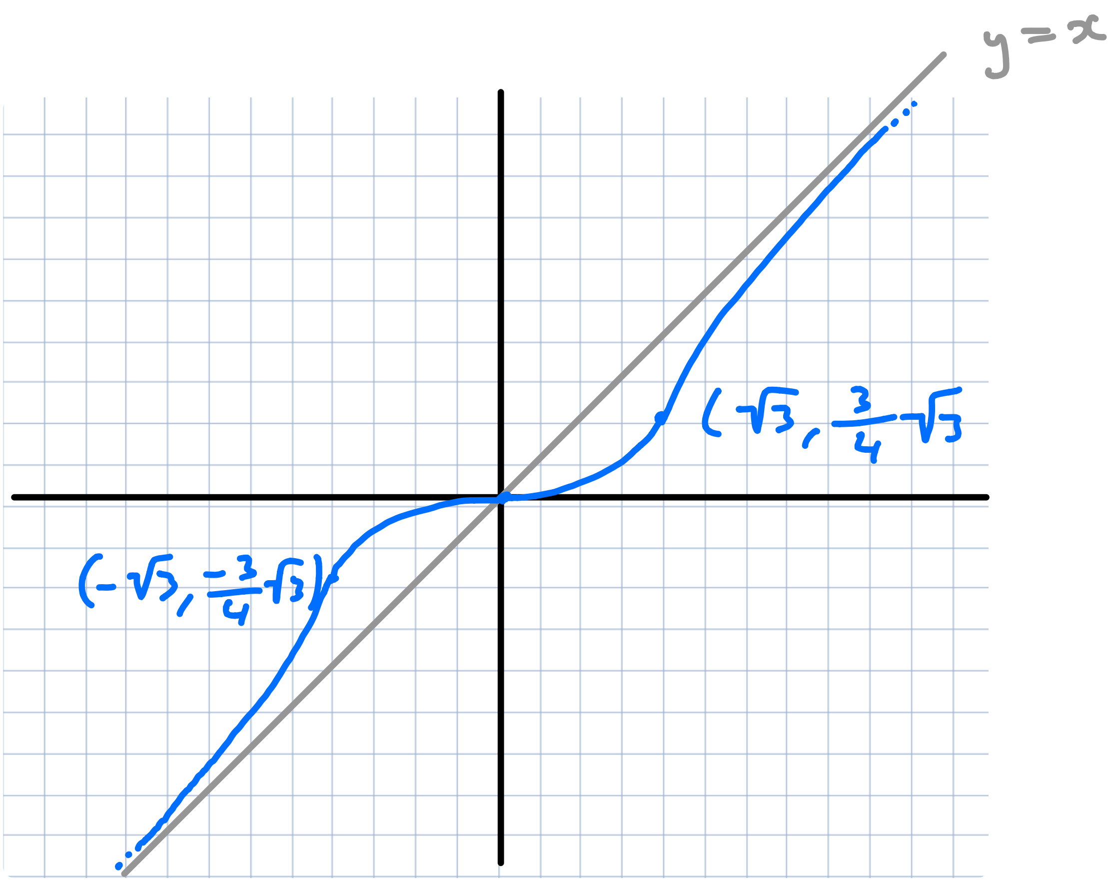
A/ $f''(x) = 0 \Leftrightarrow x = 0, \pm\sqrt{3}$

B/ None

$f(0) = 0$

$f(\sqrt{3}) = \dfrac{3\sqrt{3}}{4}$

$f(-\sqrt{3}) = \dfrac{-3\sqrt{3}}{4}$



6. (25 points) Show that the tangent line to the curve

$$y = \int_{-3x}^{x^2} te^{t^2}\,dt$$

at $x = 3$, does not contain $(0,0)$.

Solution:

$$y = \int_0^{x^2} te^{t^2}\,dt - \int_0^{-3x} te^{t^2}\,dt$$

$$\Rightarrow \frac{dy}{dx} = 2x \cdot x^2 e^{x^4} - (-3)(-3x)e^{(-3x)^2}$$

$$\Rightarrow \frac{dy}{dx} = 2 \cdot 3^3 \cdot e^{3^4} - 3^3 e^{3^4} = 3^3 e^{3^4} = 27e^{81}$$

$$y(3) = \int_{-9}^{9} te^{t^2}\,dt = 0$$

odd

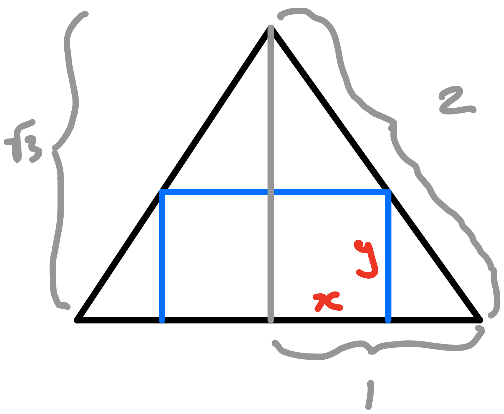
$\Rightarrow$ Tangent line goes through $(3,0)$

and has slope $27e^{81} \neq 0$

$\Rightarrow$ $(0,0)$ <u>not</u> on tangent line

7. (25 points) Find the area of the largest rectangle that can be inscribed in an equilateral triangle with sides of length 2 if one side of the rectangle lies on the base of the triangle.

Solution:



Objective : $2xy$

Constraint : $\dfrac{y}{1-x} = \dfrac{\sqrt{3}}{1}$

$\Rightarrow y = \sqrt{3}(1-x)$

$\Rightarrow 2xy = 2\sqrt{3}(x - x^2) = f(x)$

Domain $= [0,1]$ ← closed interval

$f'(x) = 2\sqrt{3} - 4\sqrt{3}x$

A/ $f'(x) = 0 \Leftrightarrow x = \frac{1}{2}$

B/ $0, 1$

$f(0) = 0$

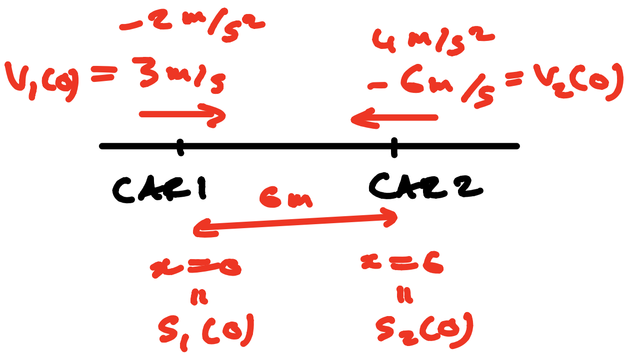
$f(1) = 0$

$f(\frac{1}{2}) = 2\sqrt{3} \cdot \frac{1}{4} = \frac{\sqrt{3}}{2}$ ← Area of largest rectangle

8. Two cars are travelling directly towards each other on a straight road. The first car is travelling at 3 metres per second. The second car is travelling at 6 metres per second. When they are 6 metres apart they simultaneously apply the brakes. The first car decelerates at a constant rate of 2 metres per second per second. The second car decelerates at a constant rate of 4 metres per second per second.

   (a) (15 points) How long after applying the brakes will the cars collide? Carefully justify your answer.

   Solution:



$a_1(t) = -2 \quad \Rightarrow \quad v_1(t) = -2t + 3$

$a_2(t) = 4 \qquad\quad v_2(t) = 4t - 6$

$\Rightarrow \; s_1(t) = -t^2 + 3t \quad (s_1(0) = 0)$

$s_2(t) = 2t^2 - 6t + 6 \quad (s_2(0) = 6)$

$s_1(t) = s_2(t) \Rightarrow -t^2 + 3t = 2t^2 - 6t + 6$

$\Rightarrow 3t^2 - 9t + 6 = 0 \Rightarrow 3(t-1)(t-2) = 0$

$\Rightarrow t = 1$ or 2

$1 < 2 \Rightarrow$ Cars collide when $t = 1$

   (b) (5 points) How far will be second car travel between applying the brakes and colliding with the first car?

   Solution:

$s_2(1) = 2$

$s_2(0) = 6$

$\Rightarrow$ The second car travels 4m between hitting the break and crashing.

9. (25 points) Calculate the following limit:

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{n+2i}{n^2+in}$$

$2(a+i\Delta x)-1$

Solution:

$\frac{b-a}{n} = \Delta x$

$$\frac{n+2i}{n^2+in} = \frac{1+\frac{2i}{n}}{1+\frac{i}{n}}\cdot\frac{1}{n}$$

$a+i\Delta x$

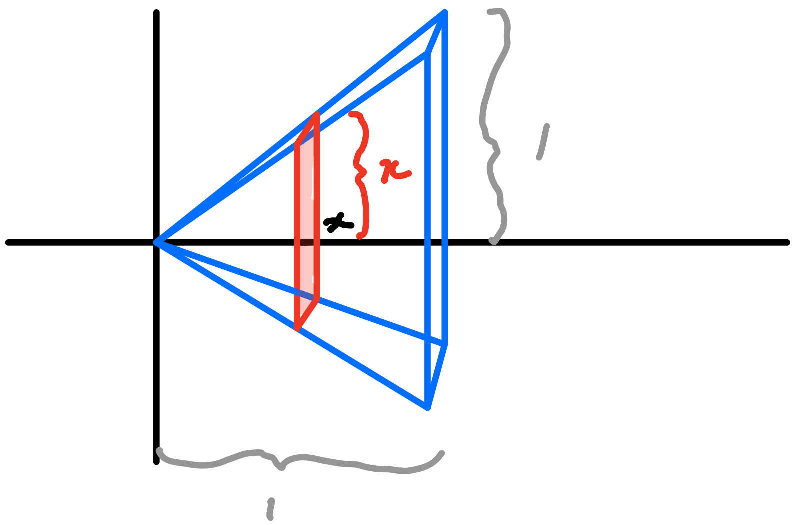
Let $a=1$, $b=2$, $f(x) = \frac{2x-1}{x} = 2-\frac{1}{x}$

$$\Rightarrow \lim_{n\to\infty}\sum_{i=1}^{n}\frac{n+2i}{n^2+in} = \int_1^2 2-\frac{1}{x}\,dx = 2x-\ln|x|\Big|_1^2$$

$$= 2-\ln(2)$$

10. (25 points) Calculate the volume of a pyramid with a square 2 by 2 base and height 1.

Solution:



$$\Rightarrow \quad A(x) = (2x)^2 = 4x^2$$

$$\Rightarrow \quad \text{Volume} = \int_0^1 4x^2\,dx = \frac{4}{3}x^3 \Big|_0^1 = \frac{4}{3}$$

END OF EXAM