MATH 1A FINAL (PRACTICE 2)
PROFESSOR PAULIN

DO NOT TURN OVER UNTIL INSTRUCTED TO DO SO.
CALCULATORS ARE NOT PERMITTED
YOU MAY USE YOUR OWN BLANK PAPER FOR ROUGH WORK
SO AS NOT TO DISTURB OTHER STUDENTS, EVERYONE MUST STAY UNTIL THE EXAM IS COMPLETE
REMEMBER THIS EXAM IS GRADED BY A HUMAN BEING. WRITE YOUR SOLUTIONS NEATLY AND COHERENTLY, OR THEY RISK NOT RECEIVING FULL CREDIT
THIS EXAM WILL BE ELECTRONICALLY SCANNED. MAKE SURE YOU WRITE ALL SOLUTIONS IN THE SPACES PROVIDED. YOU MAY WRITE SOLUTIONS ON THE BLANK PAGE AT THE BACK BUT BE SURE TO CLEARLY LABEL THEM

Name: ______________________________

Student ID: ______________________________

GSI's name: ______________________________

This exam consists of 10 questions. Answer the questions in the spaces provided.

1. Calculate the following. You do not need to simplify your answers.

(a) (10 points)

$$\frac{d}{dx}\frac{\tan(\sin(e^x))}{x}$$

Solution:

$$\frac{d}{dx}\frac{\tan(\sin(e^x))}{x} = \frac{\sec^2(\sin(e^x))\cdot\cos(e^x)\cdot e^x\cdot x - \tan(\sin(e^x))}{x^2}$$

(b) (15 points)

$$\frac{d}{dx}\cos(x)^{\sin(\sqrt{x})}$$

Solution:

$$f(x) = \cos(x)^{\sin(\sqrt{x})} \Rightarrow \ln(f(x)) = \sin(\sqrt{x})\ln(\cos(x))$$

$$\Rightarrow \frac{d}{dx}\ln(f(x)) = \cos(\sqrt{x})\cdot\frac{1}{2\sqrt{x}}\ln(\cos(x)) + \sin(\sqrt{x})\frac{-\sin(x)}{\cos(x)}$$

$$\Rightarrow \frac{d}{dx}\cos(x)^{\sin(\sqrt{x})} =$$

$$\cos(x)^{\sin(\sqrt{x})}\cdot\left(\cos(\sqrt{x})\cdot\frac{1}{2\sqrt{x}}\ln(\cos(x)) + \sin(\sqrt{x})\frac{-\sin(x)}{\cos(x)}\right)$$

2. Calculate the following (you do not need to use the (ϵ, δ)-definition):

(a) (10 points)

$$\lim_{x \to 0} \frac{\cos(2x^2) - 1}{4x}$$

Solution:

L'Hospital

$$\lim_{x \to 0} \frac{\cos(2x^2) - 1}{4x} = \lim_{x \to 0} \frac{-4x \sin(2x^2)}{4} = 0$$

(b) (15 points)

$$\lim_{x \to \infty} (1 + \frac{4}{x})^x$$

Solution:

Recall : $\lim_{z \to \infty} (1 + \frac{1}{z})^z = e$

$$\Rightarrow \lim_{x \to \infty} (1 + \frac{4}{x})^x = \lim_{x \to \infty} \left(\left(1 + \frac{1}{(\frac{x}{4})}\right)^{\frac{x}{4}} \right)^4$$

$$= \left(\lim_{x \to \infty} \left(1 + \frac{1}{(\frac{x}{4})}\right)^{\frac{x}{4}} \right)^4 = e^4$$

3. Calculate the following (you do not need to use the Riemann sum definition):

(a) (10 points)

$$\int \frac{\arccos(x)}{\sqrt{1-x^2}}dx$$

Solution:

$$u = \arccos(x) \Rightarrow \frac{du}{dx} = \frac{-1}{\sqrt{1-x^2}} \Rightarrow dx = -\sqrt{1-x^2}\,du$$

$$\Rightarrow \int \frac{\arccos(x)}{\sqrt{1-x^2}}dx = \int -u\,du = -\frac{1}{2}u^2 + C$$

$$= -\frac{1}{2}(\arccos(x))^2 + C$$

(b) (10 points)

$$\int_{1/\pi}^{2/\pi} \frac{\sin(1/x)}{3x^2}dx$$

Solution:

$$u = \frac{1}{x} \Rightarrow \frac{du}{dx} = \frac{-1}{x^2} \Rightarrow dx = -x^2 dx$$

$$\Rightarrow \int \frac{\sin(1/x)}{3x^2}dx = \frac{-1}{3}\int \sin(u)\,du = \frac{1}{3}\cos(u) + C$$

$$= \frac{1}{3}\cos(1/x) + C$$

$$\Rightarrow \int_{1/\pi}^{2/\pi} \frac{\sin(1/x)}{3x^2}dx = \frac{1}{3}\cos\left(\frac{\pi}{2}\right) - \frac{1}{3}\cos(\pi) = \frac{1}{3}$$

4. (25 points) Calculate the equation of the tangent line at $x = 3$ of the following curve:

$$\frac{2y^3(x-3)}{x\sqrt{y}} + 1 = \frac{3}{x\sqrt{y}}.$$

Solution:

$$\frac{2y^3(x-3)}{x\sqrt{y}} + 1 = \frac{1}{x\sqrt{y}} \Rightarrow 2y^3(x-3) + x\sqrt{y} = 1$$

$$\frac{d}{dx}\left(2y^3(x-3) + x\sqrt{y}\right) = \frac{d}{dx}(3)$$

$$\Rightarrow 6y^2\frac{dy}{dx}(x-3) + 2y^3 + \sqrt{y} + x\frac{1}{2}y^{-\frac{1}{2}}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\sqrt{y} - 2y^3}{6y^2(x-3) + x\frac{1}{2}y^{-\frac{1}{2}}}$$

$$x = 3 \Rightarrow 3\sqrt{y} = 3 \Rightarrow y = 1$$

$$\Rightarrow \frac{dy}{dx} = \frac{-1-2}{3\cdot\frac{1}{2}} = \frac{-3}{(3/2)} = -2$$

$\Rightarrow$ Equation of Tangent at $(3,1)$: $y - 1 = -2(x-3)$

5. (25 points) Sketch the following curve. Be sure to indicate asymptotes, local maxima and minima and concavity. Show your working on this page and draw the graph on the next page.

$$y = \ln\left(\frac{x}{2x^2 - x^3}\right)$$

Solution:

$$y = \begin{cases} -\ln(2x - x^2) & \text{if } x \neq 0 \\ \text{DNE} & \text{if } x = 0 \end{cases}$$

Not defined at 0

$\Rightarrow$ should sketch $y = f(x) = -\ln(2x - x^2)$

Domain $2x - x^2 = x(2-x) > 0 \Leftrightarrow 0 < x < 2$

Odd/Even: Neither

Vertical asymptotes: $\lim_{x \to 2^-} -\ln(2x - x^2) = \infty = \lim_{x \to 0^+} -\ln(2x - x^2)$

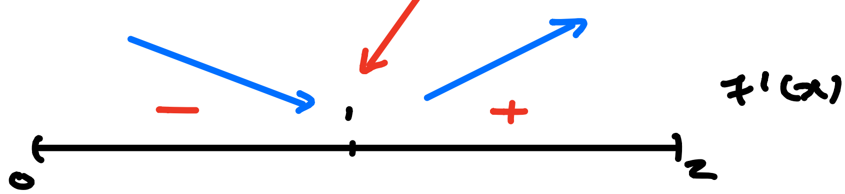
$\Rightarrow x = 0$ and $x = 2$ vertical asymptotes

$$f'(x) = \frac{2x - 2}{2x - x^2}$$

A/ $f'(x) = 0 \Leftrightarrow x = 1$

B/ None in $(0, 2)$

f cts at 1 $\Rightarrow$ local min



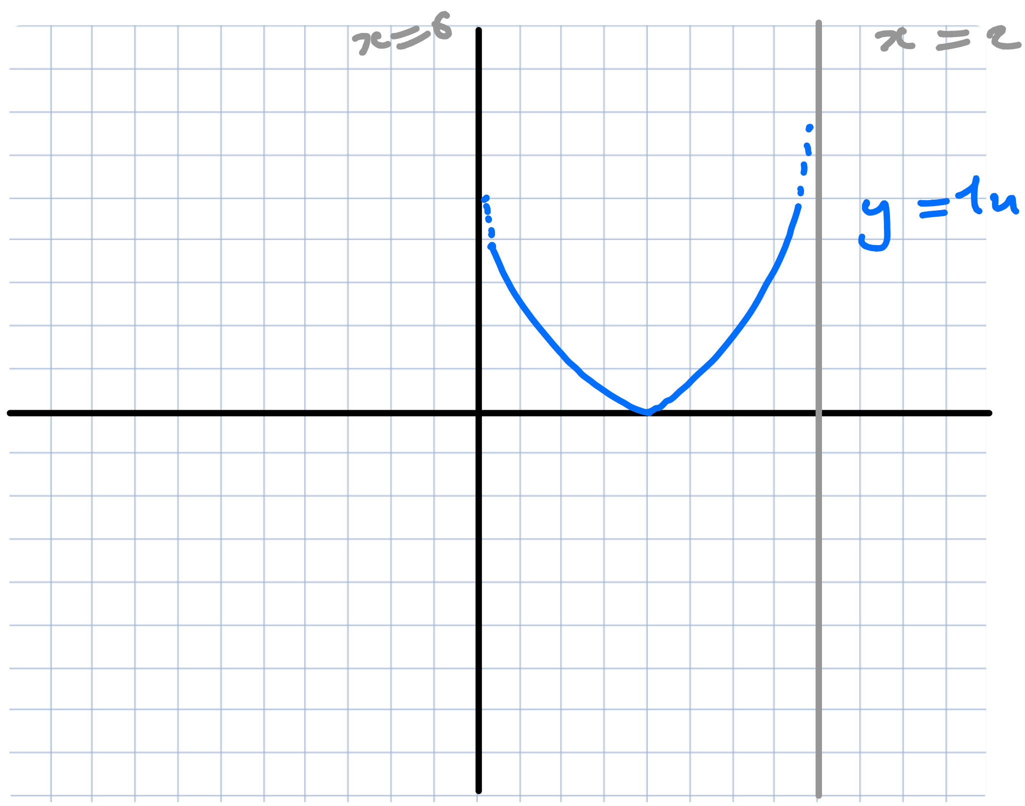
$$f''(x) = \frac{2(2x - x^2) + (2x-2)(2x-2)}{(2x - x^2)^2} = \frac{2((x-1)^2 + 1)}{(2x - x^2)^2} > 0$$

A/ None

B/ None

Solution (continued) :

$f(1) = 0$



$y = \ln\left(\frac{x}{2x^2 - x^3}\right)$

6. (25 points) Show that the following equation has at least one real solution. Be sure to carefully justify you answer clearly stating any results you use from lectures.

$$x \arctan|x| + c = 0, \text{ where } c \text{ is a constant.}$$

Hint: Consider behavior at ∞.

Solution:

Let $f(x) = x \arctan|x| + c$

$$\lim_{x \to \pm\infty} x = \pm\infty$$

$$\lim_{x \to \pm\infty} \arctan|x| = \frac{\pi}{2} > 0$$

$$\Rightarrow \lim_{x \to \infty} x \arctan|x| + c = \infty$$

$$\lim_{x \to -\infty} x \arctan|x| + c = -\infty$$

$\Rightarrow$ There exists a, b such that $f(a) < 0 < f(b)$

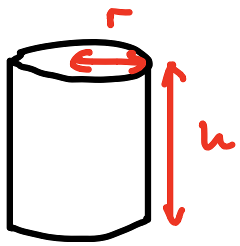
I.V.T.

$f(x)$ cts on $[a, b]$ $\Rightarrow$ There exists c such that $f(c) = 0$

7. (25 points) A drink will be packaged in a cylindrical can with volume 40in^3. The top and bottom of the can cost 4 cents per square inch. The sides cost 3 cents per square inch. Determine the dimensions of the can that minimize the cost.

Solution:

Objective : Minimize costs



cost of top and bottom

cost of sides

Objective : $4\pi r^2 + 4\pi r^2 + 3 \cdot 2\pi r h$

Constraint : $\pi r^2 h = 40$

$\Rightarrow \quad h = \dfrac{40}{\pi r^2}$

$$\Rightarrow \quad 8\pi r^2 + 6\pi r h = 8\pi r^2 + 6\pi r \cdot \frac{40}{\pi r^2}$$

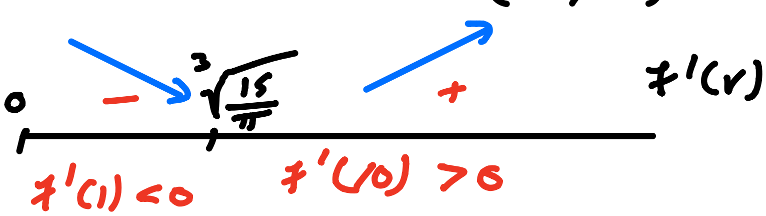
$$= 8\pi r^2 + \frac{240}{r} = f(r)$$

Domain : $(0, \infty)$

$f'(r) = 16\pi r - \dfrac{240}{r^2}$

A/ $f'(r) = 0 \Rightarrow 16\pi r = \dfrac{240}{r^2} \Rightarrow r^3 = \dfrac{\quad}{\pi} \Rightarrow r = \sqrt[3]{\dfrac{15}{\pi}}$

B/ f' continuous on $(0, \infty)$



$\Rightarrow$ Absolute min cost is when $r = \sqrt[3]{\dfrac{15}{\pi}}$, $h = \dfrac{40}{\pi \left(\frac{15}{\pi}\right)^{\frac{2}{3}}}$

8. Two rockets are fired vertically into the air from the ground. The second rocket is launched four seconds after the first. The velocity of the first rocket is $v_1(t) = 6 - t$ metres per second and the velocity of the second is $v_2(t) = 10 - t$ metres per second, where t is the time in seconds after the first launch.

(a) (15 points) How long after first the first launch will both rockets be at the same height? What will this height be?

Solution:

$s_1(t) = 6t - \frac{1}{2}t^2 + C$ and $s_1(0) = 0 \Rightarrow C = 0$

$\Rightarrow s_1(t) = 6t - \frac{1}{2}t^2$

$s_2(t) = 10t - \frac{1}{2}t^2 + C$ and $s_2(4) = 0 \Rightarrow 40 - 8 + C = 0$

$\Rightarrow C = -32 \Rightarrow s_2(t) = 10t - \frac{1}{2}t^2 - 32$

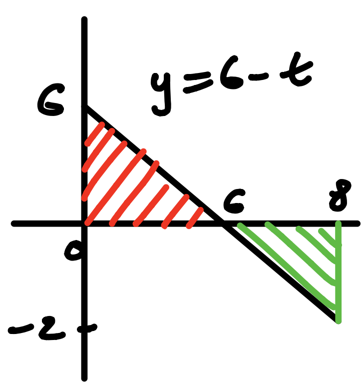
$s_1(t) = s_2(t) \Rightarrow 6t - \frac{1}{2}t^2 = 10t - \frac{1}{2}t^2 - 32$

$\Rightarrow 4t = 32 \Rightarrow t = 8. \leftarrow$ time after 1st lauch they'll be same height

$s_1(8) = s_2(8) = 16 \leftarrow$ height at $t=8$

(b) (10 points) Determine the total distance traveled by the first rocket at this time.

Solution:



Total Distance travelled

= Area + Area

$= \frac{6 \times 6}{2} + \frac{2 \times 2}{2} = 18 + 2 = 20$ m.

9. (25 points) Calculate the following limit:

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{n}{n^2+3i^2}$$

Hint: Remember that $\frac{A}{B}=\frac{1}{B/A}$.

Solution:

$$\frac{n}{n^2+3i^2} = \frac{1}{\frac{n^2+3i^2}{n^2}}\cdot\frac{1}{n} = \frac{1}{1+3\left(\frac{i}{n}\right)^2}\cdot\frac{1}{n}$$

Letting $a=0$, $b=1$, $f(x)=\frac{1}{1+3x^2}$ we get

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{n}{n^2+3i^2} = \int_0^1 \frac{1}{1+3x^2}\,dx$$

Let $u=\sqrt{3}x \Rightarrow \frac{du}{dx}=\sqrt{3} \Rightarrow dx=\frac{1}{\sqrt{3}}du$

$$\Rightarrow \int\frac{1}{1+3x^2}\,dx = \frac{1}{\sqrt{3}}\int\frac{1}{1+u^2}\,du = \frac{1}{\sqrt{3}}\arctan(u)+C$$

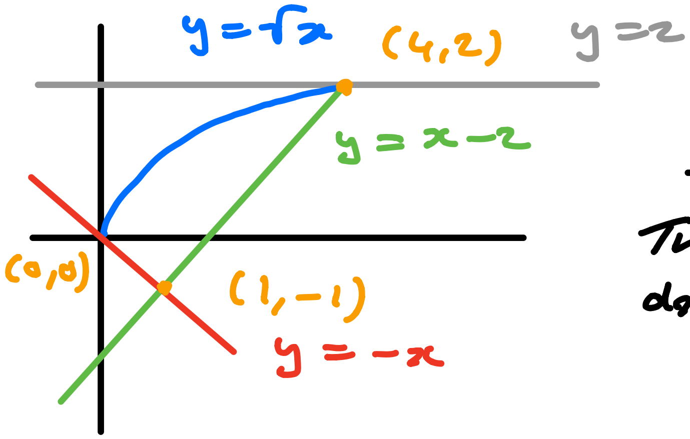
$$= \frac{1}{\sqrt{3}}\arctan(\sqrt{3}x)+C$$

$\Rightarrow$

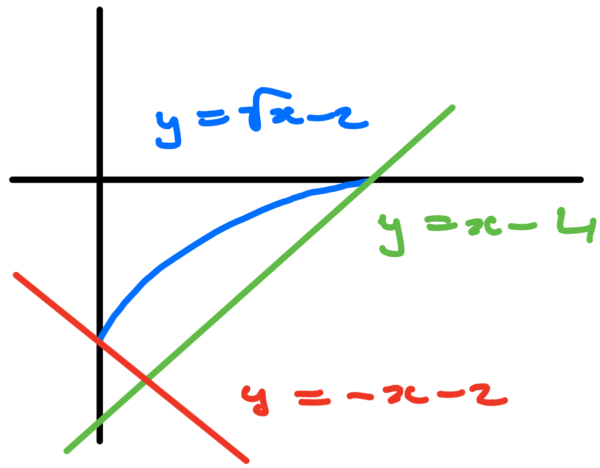
$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{n}{n^2+3i^2} = \frac{1}{\sqrt{3}}\arctan(\sqrt{3}) = \frac{\pi}{\sqrt{3}\cdot 3}$$

10. (25 points) Calculate the volume of the solid of revolution formed by rotating the region enclosed by $x = y^2$ (with $y \geq 0$), $x = -y$ and $x = y + 2$ around the line $y = 2$.

Solution:



$\longrightarrow$ Translate down by 2



$$\Rightarrow \text{Volume} = \pi \int_0^1 (-x-2)^2 - (\sqrt{x} - 2)^2 \, dx + \pi \int_1^4 (x-4)^2 - (\sqrt{x} - 2)^2 \, dx$$

$$= \pi \int_0^1 (-x-2)^2 \, dx + \pi \int_1^4 (x-4)^2 \, dx - \pi \int_0^4 x - 4\sqrt{x} + 4 \, dx$$

$$= \frac{\pi}{3} (x+2)^3 \Big|_0^1 + \frac{\pi}{3} (x-4)^3 \Big|_1^4 - \pi \left(\frac{1}{2} x^2 - \frac{8}{3} x^{3/2} + 4x \right) \Big|_0^4$$

$$= \left(9\pi - \frac{8\pi}{3} \right) + (0 - (-9\pi)) - 8\pi + \frac{64}{3}\pi - 16\pi$$

$$= \frac{38}{3}\pi$$

END OF EXAM